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A new analytical technique for strongly nonlinear damped forced systems

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Abstract Combining the general Struble's technique and homotopy perturbation method, an analytical technique has been presented to determine approximate solutions of strongly nonlinear differential systems with severe damping effect in the presence of an external force. The method is illustrated by examples. The approximate solutions show a good coincidence with corresponding numerical solution (considered to be exact). Moreover, it provides better result than other existing solutions (derived by several analytical methods).

1. Introduction

Krylov–Bogoliubov-Mitropolskii (KBM) [1,2] method is one of the most widely used methods to obtain the approximate solutions of weakly nonlinear differential systems. Popov [3] extended the method to nonlinear damped oscillatory systems. Bojadziev et al. [4] presented a comparison of Poincare [5] and KBM methods. Shamsul [6] developed a general Struble's technique to determine approximate solution of an n-th order weakly nonlinear differential system. The results of Shamul's [6] are similar to those obtained by extended KBM method (by Popov [3]). Bojadziev [7] studied a damped forced weakly nonlinear system. Shamsul [8] extended KBM method for solving an n-th order time dependent nonlinear differential system in which an external periodic force acts. Some weakly nonlinear systems were also studied in [9–15]. Nagy and Balachandran [16] utilized perturbation method to investigate jump phenomena of weakly nonlinear systems in which a weak damping force acts. Recently, strong nonlinear differential systems have been investigated in [17–21]. Lakrad and Belhaq [22] used multiple scales technique to find periodic solutions of strongly nonlinear oscillators for free vibration. Pakdemirli et al. [23] used multiple scales Lindstedt-Poincare (MSLP) to determine approximate solution for strongly nonlinear damped system in the presence of an external force. He [24] developed homotopy perturbation method to solve strongly nonlinear systems. But it [24] is useless when the damping or/and external forces act on the systems. On the contrary, the classical perturbation methods [1–16] are valid only for weakly nonlinear systems (see [8] for details). Thus the combination of two methods (perturbation method and homotopy perturbation method) is needed to handle the strongly nonlinear differential systems with strong damping effect. In this article, combination of the general Struble's technique [6] and homotopy perturbation method [25,26] has been

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utilized to obtain approximate solutions of strongly nonlinear problems with severe damping effect. The solution nicely agrees with the numerical solution. It is noted that, the solution obtained in [23] is valid for the strong nonlinear problems with small damping effects.

2. The method

Let us consider a second order time dependent strongly nonlinear, non-autonomous differential system

$$\ddot{x} + 2k\dot{x} + \omega_0^2 x = \varepsilon f(x, \dot{x}) + \varepsilon_1 E\Phi(\nu t), \tag{1}$$

where over dot denotes the differentiation with respect to t and $\omega_0 \geqslant 0$, k, E, ν are constants. Herein ε_1 and ε denote small and large parameters respectively, ω_0 is the natural frequency and $\Phi(\nu t)$ is a periodic function. The nonlinear function, $f(x, \dot{x})$ satisfies the condition $f(-x, -\dot{x}) = -f(x, \dot{x})$.

In the previous articles [23,27], k (damping constant) was considered small but for large values of k, the solutions obtained in [23,27] do not provide desire results. The main exclusive aim of this article is to remove this limitation.

In order to use homotopy perturbation method, we may rewrite Eq. (1) in the form

$$\ddot{x} + 2k\dot{x} + \omega^2 x = (\omega^2 - \omega_0^2)x + \varepsilon f(x, \dot{x}) + \varepsilon_1 E\Phi(\nu t), \tag{2}$$

where ω is an unknown frequency of the oscillator.

For Eq. (2) we can establish the following homotopy:

$$\ddot{x} + 2k\dot{x} + \omega^2 x = p[(\omega^2 - \omega_0^2)x + \varepsilon f(x, \dot{x}) + \varepsilon_1 E\Phi(\nu t)], \tag{3}$$

where p is the homotopy parameter.

When $p = 0$, Eq. (3) becomes a linear equation and it has two eigen-values, say $\lambda_1 = -k + iq$, $\lambda_2 = -k - iq$, $q = \sqrt{\omega^2 - k^2}$, $k < \omega$. Thus the unperturbed solution becomes $x(t; 0) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}$ where a_1 and a_2 are constants.

On the other hand, Eq. (3) becomes original Eq. (1), when $p = 1$.

When $p \neq 0$, the approximate solution of Eq. (3) takes the form (see [6] for details)

$$x(t; p) = a_1(t)e^{\lambda_1 t} + a_2(t)e^{\lambda_2 t} + pu_1(a_1, a_2, t) + \ldots. \tag{4}$$

Now, Eq. (3) can be written in the following form

$$(D - \lambda_1)(D - \lambda_2)x = p[(\omega^2 - \omega_0^2)x + \varepsilon f + \varepsilon_1 E\Phi(\nu t)], \tag{5}$$

where $D = d/dt$, $\lambda_1 = -k + iq$, $\lambda_2 = -k - iq$, $q = \sqrt{\omega^2 - k^2}$, $k < \omega$.

It can be considered that a_1 and a_2 are time-dependent functions (rather than constants) on the left-hand side of Eq. (5).

Substituting Eq. (4) into Eq. (5), we obtain

$$(D - \lambda_1)(D - \lambda_2)(a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + pu_1 + \ldots)$$
$$= p[(\omega^2 - \omega_0^2)(a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + pu_1 + \ldots) + \varepsilon f + \varepsilon_1 E\Phi(\nu t)]$$

or

$$(D - \lambda_2)(\dot{a}_1 e^{\lambda_1 t}) + (D - \lambda_1)(\dot{a}_2 e^{\lambda_2 t}) + (D - \lambda_1)(D - \lambda_2)(pu_1) + \ldots$$
$$= p[(\omega^2 - \omega_0^2)(a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + pu_1 + \ldots) + \varepsilon f + \varepsilon_1 E\Phi(\nu t)], \tag{6}$$

since $(D - \lambda_1)(a_1 e^{\lambda_1 t}) = \dot{a}_1 e^{\lambda_1 t}$ and $(D - \lambda_2)(a_2 e^{\lambda_2 t}) = \dot{a}_2 e^{\lambda_2 t}$.

Let us consider the terms through $O(p)$. Then, Eq. (6) becomes

$$(D - \lambda_2)(\dot{a}_1 e^{\lambda_1 t}) + (D - \lambda_1)(\dot{a}_2 e^{\lambda_2 t}) + (D - \lambda_1)(D - \lambda_2)(pu_1)$$
$$= p[(\omega^2 - \omega_0^2)(a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}) + \varepsilon f + \varepsilon_1 E\Phi(\nu t)]. \tag{7}$$

Herein the nonlinear function f can be expanded in a Taylor series as $f = \sum_{m_1=0,\ m_2=0}^{\infty,\ \infty} F_{m_1,m_2} e^{(m_1\lambda_1 + m_2\lambda_2)t}$ and the unknown function u_1 can be found in terms of the variables a_1, a_2 and t, under the restriction that u_1 excludes the term $F_{m_1,m_2} e^{(m_1\lambda_1 + m_2\lambda_2)t}$ of f when, $m_1 - m_2 \neq \pm 1$. On the other hand, $\dot{a}_1$ and $\dot{a}_2$ respectively, contain the term $F_{m_1,m_2} e^{(m_1\lambda_1 + m_2\lambda_2)t}$ when $m_1 - m_2 = 1$ and $m_1 - m_2 = -1$. This assumption takes u_1 free from secular terms, *i.e.* and $t\cos t$, $t\sin t$. It is clear that the first term of the left side of Eq. (7) contains a term with $e^{\lambda_1 t}$ and the second term f of the right side of the same equation contains the term $e^{(m_1\lambda_1 + m_2\lambda_2)t}$. Since, $\lambda_1 = -k + iq$, $\lambda_2 = -k - iq$, then $e^{\lambda_1 t}$ and $e^{(m_1\lambda_1 + m_2\lambda_2)t}$, respectively become $e^{(-k+iq)t}$ and $e^{(-k(m_1+m_2)+iq)t}$, where $m_1 - m_2 = 1$. We observe that both $e^{\lambda_1 t}$ and $e^{(m_1\lambda_1 + m_2\lambda_2)t}$ are related to e^{iqt}, and the terms with $e^{\lambda_1 t}$ and $e^{(m_1\lambda_1 + m_2\lambda_2)t}$ of Eq. (7) are equated. In a similar way, we equate the terms with $e^{\lambda_2 t}$ and $e^{(m_1\lambda_1 + m_2\lambda_2)t}$, where $m_1 - m_2 = -1$ of Eq. (7). On other hand, the third term of the right side of Eq. (7) contains periodic function $\Phi(\nu t)(e^{i\nu t}, e^{-i\nu t})$. Since $e^{i\nu t} - e^{iqt} = O(\varepsilon_1)$ and $e^{-i\nu t} - e^{-iqt} = O(\varepsilon_1)$, two right-handed terms, $e^{i\nu t}$ and $e^{-i\nu t}$ of Eq. (7) will be added respectively, to the equations of $\dot{a}_1$ and $\dot{a}_2$. But u_1 never contains the terms $e^{i\nu t}$ and $e^{-i\nu t}$, when $\nu - 1 = O(\varepsilon_1)$ (see[6] for details).

Now, equating the coefficients of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ on both sides of Eq. (7), the following variational equations can be obtained as

$$(D - \lambda_2)(\dot{a}_1 e^{\lambda_1 t}) = p[(\omega^2 - \omega_0^2)a_1 e^{\lambda_1 t} + \varepsilon \sum_{m_1=0,\ m_2=0}^{\infty,\ \infty} F_{m_1,m_2} e^{(m_1\lambda_1 + m_2\lambda_2)t}$$
$$+ \varepsilon_1 E e^{i\nu t}],\ m_1 - m_2 = 1, \tag{8}$$

$$(D - \lambda_1)(\dot{a}_2 e^{\lambda_2 t}) = p[(\omega^2 - \omega_0^2)a_2 e^{\lambda_2 t}$$
$$+ \varepsilon \sum_{m_1=0,\ m_2=0}^{\infty,\ \infty} F_{m_1,m_2} e^{(m_1\lambda_1 + m_2\lambda_2)t}$$
$$+ \varepsilon_1 E e^{-i\nu t}],\ m_1 - m_2 = -1, \tag{9}$$

This leaves the following perturbational equation:

$$(D - \lambda_1)(D - \lambda_2)u_1 = \sum_{m_1=0,\ m_2=0}^{\infty,\ \infty} \varepsilon F_{m_1,m_2} e^{(m_1\lambda_1 + m_2\lambda_2)t},\quad m_1 - m_2 \neq \pm 1. \tag{10}$$

In order to obtain the first approximate solution, it can be considered that a_1 and a_2 are constants in the right-hand sides of Eqs. (8)–(10); so that the particular solutions of Eqs. (8) and (9) are

$$(\ddot{a}_1 + (\lambda_1 - \lambda_2)\dot{a}_1)e^{\lambda_1 t} = p[(\omega^2 - \omega_0^2)a_1 e^{\lambda_1 t}$$
$$+ \varepsilon \sum_{m_1=0,\ m_2=0}^{\infty,\ \infty} F_{m_1,m_2} e^{(m_1\lambda_1 + m_2\lambda_2)t} + \varepsilon_1 E e^{i\nu t}], \tag{11}$$

$$(\ddot{a}_2 + (\lambda_2 - \lambda_1)\dot{a}_2)e^{\lambda_2 t} = p[(\omega^2 - \omega_0^2)a_2 e^{\lambda_2 t}$$
$$+ \varepsilon \sum_{m_1=0,\ m_2=0}^{\infty,\ \infty} F_{m_1,m_2} e^{(m_1\lambda_1 + m_2\lambda_2)t} + \varepsilon_1 E e^{-i\nu t}]. \tag{12}$$

The particular solutions of Eq. (10) are given by