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On the decay of a sawtooth profile in non-ideal magneto-gasdynamics

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Abstract In the present paper, an asymptotic approach is used to analyse the main features of weakly nonlinear waves propagating in a compressible, inviscid, nonideal gas in the presence of magnetic field. An evolution equation, which characterizes the wave process in the high frequency domain and points out the possibility of wave breaking at a finite time, is derived. The growth equation governing the behaviour of an acceleration wave is recovered as a special case. Further, we consider a sufficiently weak shock at the outset and study the propagation of the disturbance given in the form of a sawtooth profile. It is observed that the non-idealness of the gas causes an early decay of the sawtooth wave as compared to ideal case however the presence of magnetic field causes to slow down the decay process as compared to non-ideal non-magnetic case. A remarkable difference in wave profiles for planar and cylindrically symmetric flows has been observed. The effect of non-idealness, in the presence of magnetic field, on the formation of shock is more dominant in case of cylindrical symmetry as compared to planar case.

1. Introduction

Discontinuity waves, also known as shock waves, acceleration waves and weak waves are characterized by discontinuity in the normal derivative of the flow variable rather than the variable itself. Therefore, for nonlinear systems, the analysis of these waves has been the subject of great interest both from mathematical and physical point of view. For the physical phenomenon modelled by a system of quasi linear hyperbolic partial differential equations, it is theoretically possible to find the progressive wave solution. Choquet-Bruhat [1] used the perturbation method to determine a shockless solution of system of quasi linear hyperbolic partial differential equations that depend upon single phase function. Germain [2], Fusco [3], Fusco and Engelbrecht [4], and Sharma et al. [5], used the same technique to analyse the nonlinear wave propagation in various gasdynamic regimes. Hunter and Keller [6] presented a method, known as ray method, to determine a small-amplitude high frequency wave solution of hyperbolic system. Jena and Singh [7] studied the problem of evolution of an acceleration wave and a characteristic shock for the system of partial differential equations describing one dimensional, unsteady, axisymmetric motion of transient pinched plasma. A detailed discussion on the method and application

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of asymptotic expansions can be seen in Miller [8] and Sharma [9]. Singh et al. [10] used perturbation scheme to study the propagation of weak shock waves in non-uniform radiative magnetogasdynamics. Singh et al. [11] have studied the problem of propagation of acceleration waves along the characteristic path by using the characteristics of the governing system as the reference coordinate system. Arora et al. [12] used the method of multiple timescales to obtain the asymptotic solutions to the planar and non-planar flows into a non-ideal gas. Sharma and Venkatraman [13] studied the asymptotic decay laws for planar and non-planar shock waves and the first order associated discontinuities that catch up with the shock.

In the present work, we deal with the study of propagation of weakly nonlinear waves in a nonideal gas permeated by a transverse magnetic field with infinite electrical conductivity. An evolution equation, characterizing the wave process in the high frequency domain, is derived. The growth equation for an acceleration wave is recovered as a special case. The propagation of a sawtooth profile that ends in a tail shock can be analysed in similar manner.

2. Governing equations

The fundamental equations for one dimensional unsteady motion of a non-ideal gas in the presence of a transverse magnetic field may be written as [14–16].

$$\rho_t + v\rho_x + \rho v_x + \rho m v x^{-1} = 0, \tag{1}$$

$$v_t + v v_x + \rho^{-1}(p_x + h_x) = 0, \tag{2}$$

$$p_t + v p_x + \rho d^2 (v_x + m v x^{-1}) = 0, \tag{3}$$

$$h_t + v h_x + 2h(v_x + m v x^{-1}) = 0, \tag{4}$$

where ρ is the density, v is the fluid velocity, p is the pressure, $d = (\gamma p/\rho(1 - b\rho))^{1/2}$ is the speed of sound in non-ideal gas with γ as the adiabatic index, b is the Van der Wall's constant, $h = \mu H^2/2$ is the magnetic pressure with H as the magnetic field strength, μ is the magnetic permeability, t is the time, and x is the spatial coordinate. Here subscripts denote partial differentiation unless stated otherwise. The letter m takes values 0 for planar and 1 for cylindrically symmetric motion.

In matrix notation, Eqs. (1)–(4) can be written as

$$U_t + A U_x + B = 0, \tag{5}$$

where

$$U = \begin{bmatrix} \rho \\ v \\ p \\ h \end{bmatrix}, \quad A = \begin{bmatrix} v & \rho & 0 & 0 \\ 0 & v & \rho^{-1} & \rho^{-1} \\ 0 & \frac{\gamma p}{1-b\rho} & v & 0 \\ 0 & 2h & 0 & v \end{bmatrix}, \quad B = \begin{bmatrix} \rho m v x^{-1} \\ 0 \\ \frac{\gamma p}{1-b\rho} m v x^{-1} \\ 2hmvx^{-1} \end{bmatrix}. \tag{6}$$

Eq. (5) can be written as

$$U^i_t + A^{ij} U^j_x + B^i = 0, \quad i,j = 1,2,3,4, \tag{7}$$

where U^i, A^{ij}, and B^i are components of column vector U, matrix A and column vector B respectively.

The system of Eq. (7) is hyperbolic and eigenvalues of the coefficient matrix A are $v - c, v, v$ and $v + c$. Here $c = (d^2 + e^2)^{1/2}$ is the magneto sonic speed with $d = (\gamma p/\rho(1 - b\rho))^{1/2}$ as the speed of sound in nonideal gas and $e = (2h/\rho)^{1/2}$ the Alfvén speed. The left and right eigenvectors of A corresponding to the eigenvalue $v + c$ are

$$l = (0, \rho c, 1, 1), r^T = (1, c/\rho, d^2, e^2), \tag{8}$$

where a superscript means transposition.

3. Progressive wave solution

Let us consider the asymptotic solution of Eq. (7) which exhibits the feature of progressive waves. Consider the following asymptotic expansion

$$U^i(x,t) = U^i_0 + \varepsilon U^i_1(x,t,\xi) + O(\varepsilon^2), \tag{9}$$

where U^i_0 is a known constant solution of Eq. (7) such that $B^i(U_0) = 0$. The remaining terms of Eq. (9) are of progressive wave nature. The choice of ε depends upon the physical problem to be studied. Let τ_{ch} be the characteristic timescale for the medium and τ_a be the attenuation time, then we define a parameter $\varepsilon = \tau_{ch}/\tau_a \ll 1$. The variable ξ is a "fast variable" defined as $\xi = f(x,t)/\varepsilon$, where $f(x,t)$ is a phase function to be determined later. It may be noticed that the case $\varepsilon \ll 1$, which corresponds to the situation in which the characteristic frequency of the medium is very large than the attenuation frequency of the signal, characterizes a high frequency propagation [17].

Introducing the Taylor's series expansion of A^{ij} and B^i in the neighbourhood of the known constant solution U^i_0 and using Eq. (9), we get

$$A^{ij} = A^{ij}_0 + \varepsilon \left(\frac{\partial A^{ij}}{\partial U^k} \right)_0 U^k_1 + O(\varepsilon^2), \tag{10}$$

$$B^i = B^i_0 + \varepsilon \left(\frac{\partial B^i}{\partial U^k} \right)_0 U^k_1 + O(\varepsilon^2). \tag{11}$$

Substituting Eqs. (9)–(11) in Eq. (7) and cancelling the coefficient of ε^0 and ε^1, we get

$$\left(A^{ij}_0 - \lambda \delta^i_j \right) \frac{\partial U^j_1}{\partial \xi} = 0, \tag{12}$$

$$\left(A^{ij}_0 - \lambda \delta^i_j \right) \frac{\partial U^j_2}{\partial \xi} + \left(\frac{\partial U^i_1}{\partial t} + A^{ij}_0 \frac{\partial U^j_1}{\partial x} \right) f_x^{-1} + U^k_1 \left(\frac{\partial A^{ij}}{\partial U^k} \right)_0 \frac{\partial U^j_1}{\partial \xi} + f_x^{-1} U^k_1 \left(\frac{\partial B^i}{\partial U^k} \right)_0 = 0, \tag{13}$$

where $\lambda = -f_t/f_x$, δ^i_j is the Krönecker delta and the subscript 0 means the quantity involved is evaluated at constant state U_0. Eq. (12) yields the characteristic polynomial $\lambda^2(\lambda^2 - c_0^2) = 0$, providing nonzero eigenvalues $\pm c_0$ of A_0. Considering the velocity $\lambda = c_0$ the corresponding left and right eigenvectors of A_0 are given by Eq. (8) with subscript 0. From Eq. (12) we see that $\partial U_1/\partial \xi$ is collinear to r_0 and therefore U_1 may be written as

$$U_1(x,t,\xi) = \alpha(x,t,\xi) r_0 + W(x,t), \tag{14}$$

representing a solution of Eq. (12). Here $\alpha(x,t,\xi)$ is the amplitude factor to be determined and the W^i (the components of the column vector W) are integration constants which are not of progressive wave nature and therefore can be taken as zero. Now the phase function $f(x,t)$ is determined by