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Orthogonal double cover of Complete Bipartite Graph by disjoint union of complete bipartite graphs

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Abstract Let H be a graph on n vertices and $\mathcal{G}$ a collection of n subgraphs of H, one for each vertex, $\mathcal{G}$ is an *orthogonal double cover* (ODC) of H if every edge of H occurs in exactly two members of $\mathcal{G}$ and any two members share an edge whenever the corresponding vertices are adjacent in H and share no edges whenever the corresponding vertices are nonadjacent in H. In this paper, we are concerned with symmetric starter vectors of the orthogonal double covers (ODCs) of the complete bipartite graph and using the method of cartesian product of symmetric starter vectors to construct ODC of the complete bipartite graph by G, where G is a complete bipartite graph, disjoint union of different complete bipartite graphs and disjoint union of finite copies of a complete bipartite graph.

1. Introduction

All graphs here are, finite, simple and undirected. Let H be any regular graph and let $\mathcal{G} = \{G_0, G_1, \ldots, G_{|V(H)|-1}\}$ be a collection of $|V(H)|$ subgraphs (pages) of H. $\mathcal{G}$ is an orthogonal double cover (ODC) of H if it has the following properties.

i. Double cover property.Every edge of H is contained in exactly two of the pages in $\mathcal{G}$.

ii. Orthogonality propertyFor any two distinct pages G_i and G_j in $\mathcal{G}|E(G_i) \cap E(G_j)| = 1$ if and only if i and j are adjacent in H.

If all pages $G_i \cong G$ for all $i \in \{0, 1, \ldots, |V(H)| - 1\}$, then $\mathcal{G}$ is an ODC of H by G. For the definition of an orthogonal double cover (ODC) of the complete graph K_n by a graph G and for a survey on this topic, see [1]. While in principle any regular graph is worth considering. The choice of $H = K_{n,n}$ is quite natural, also in view of a technical motivation: ODCs of such graphs are a helpful tool for constructing ODCs of K_n (see [4]).

In this paper, we assume $H = K_{n,n}$ the complete bipartite graph with partition sets of size n each. Furthermore we make use of the usual notation: $D \cup F$ for the disjoint union of D and F and mD for m disjoint copies of D.

Denote the vertices of the partition sets of $K_{n,n}$ by $\{0_0, 1_0, \ldots, (n-1)_0\}$ and $\{0_1, 1_1, \ldots, (n-1)_1\}$. The length of an edge $\{x_0, y_1\}$ of $K_{n,n}$ is defined to be the difference y-x, where $xy \in \mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$. Note that sums and differences

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are calculated in $\mathbb{Z}_n$ (that is, sums and differences are calculated modulo n). If there is no danger of ambiguity if $\{x_0, y_1\} \in E(K_{n,n})$ we can write $\{x_0, y_1\}$ as x_0y_1.

Let G be a subgraph of $K_{n,n}$ and $a \in \mathbb{Z}_n$. The *a-translate* of G, denoted by $G + a$ is the edge-induced subgraph of $K_{n,n}$ induced by $\{(x + a)_0(y + a)_1{:}x_0y_1 \in E(G)\}$. A subgraph G of $K_{n,n}$ is called *half-starter* if $|E(G)| = n$ and the lengths of all edges in G are mutually different. We denote a half-starter G by the vector $v(G) = (v_0, v_1, \ldots, v_{n-1})$, where $v_0, v_1, \ldots, v_{n-1} \in \mathbb{Z}_n$ and v_i can be obtained from the unique edge $(v_i)_0(v_i + i)_1$ of length i in G. Two half-starters $v(G) = (v_0, v_1, \ldots, v_{n-1})$ and $v(F) = (u_0, u_1, \ldots, u_{n-1})$ are said to be orthogonal if $\{v_i - u_i : i \in \mathbb{Z}_n\} = \mathbb{Z}_n$. For a subgraph G of $K_{n,n}$ with n edges, the edge-induced subgraph G_s with $E(G_s) = \{y_0x_1{:}x_0y_1 \in E(G)\}$ is called the symmetric graph of G.

El-Shanawany et al. [4] established following three theorems.

Theorem 1.1. *If G is a half-starter, then the collection of all translates of G forms an edge-decomposition of $K_{n,n}$.*

Theorem 1.2. *If two half-starter v(G) and v(F) are orthogonal, then the union of the set of translates of G and the set of translates of F forms an ODC of $K_{n,n}$.*

Theorem 1.3. *A half-starter is a symmetric starter if and only iff*

$\{v_i - v_{-i} + i : i \in \mathbb{Z}_n\} = \mathbb{Z}_n$.

An algebraic construction of ODCs via symmetric starters has been exploited to get a complete classification of ODCs of $K_{n,n}$ by G for $n \leqslant 9$: a few exceptions apart, all graphs G are found this way (see [4]). Much of research on this subject focused with the detection of ODCs with pages isomorphic to a given graph G. For more results on ODCs, see [2,7]. In [3], the other terminologies not defined here can be found.

In constructing ODCs a natural approach is to try to put two given ODCs somehow together to obtain ODCs of a larger complete bipartite graph. The following theorem of El-Shanawany et al. [6] relates the ODCs of complete bipartite graphs and the Cartesian product of two symmetric starter vectors.

Theorem 1.4. *[6] The Cartesian product of any two symmetric starter vectors is a symmetric starter vector with respect to the Cartesian product of the corresponding groups.*

It should be noted that $v(D) \times v(F)$ is not the usual cartesian product of the graphs D and F that has been studied widely in the literature. Our results based on the following symmetric starter vectors for a few classes of graphs that can be used as ingredients for cartesian product construction to obtain new symmetric starter vectors of an ODC of the complete bipartite graph by disjoint union of complete bipartite graphs.

I. nK_2 is a symmetric starter vector of an ODC of $K_{n,n}$ and $v(nK_2) = (0, 1, 2, \ldots, n-1)$ where $n \equiv 1,5 \bmod 6$, see [4].

II. $K_{1,n}$ is a symmetric starter vector of an ODC of $K_{n,n}$ and $v(K_{1,n}) = (c, c, \ldots, c)$ where $\in \mathbb{Z}_n$, see [4].

III. $K_{r,s}$ is a symmetric starter vector of an ODC of $K_{n,n}$ and

$$v(K_{r,s}) = \left(\overbrace{0,0,\ldots,0}^{s\text{ times}}, \overbrace{s(r-1), s(r-1), \ldots, s(r-1)}^{s\text{ times}}, \cdots, \overbrace{s,s,\ldots,s}^{s\text{ times}}\right)$$

where $n = rs$, See [5].

IV. $2K_{1,n}$ is a symmetric starter vector of an ODC of $K_{2n,2n}$ and $v(2K_{1,n}) = (n, n-2, n, n-2, \ldots, n, n-2)$ where $n \geqslant 2$ see [5].

V. $K_{1,4} \cup K_{2,1} \cup K_{1,1} \cup K_{1,2n\text{-}6}$ is a symmetric starter vector of an ODC of $K_{2n+1,\,2n+1}$ and $v(K_{1,4} \cup K_{2,1} \cup K_{1,1} \cup K_{1,2n-6}) = (3, 2n, 2n, 2n-3, 0, 0, 0, \ldots, 0, 0, 0, 2, 2n, 2n)$ where $n \geqslant 4$ see [5].

2. ODCs of $K_{mn,mn}$ by complete bipartite graphs

In the following, if there is no danger of ambiguity, we can write (a, b) as ab, if $(a, b) \in \mathbb{Z}_m \times \mathbb{Z}_n$ (the Cartesian product of $\mathbb{Z}_m$ and $\mathbb{Z}_n$). Furthermore, we make the use of A for a subset of $\mathbb{Z}_m$, where, $A = \{x{:}x = s(r-j)$ for all $1 \leqslant j \leqslant r\}$ for a positive integer m such that $m = rs$.

Theorem 2.1. *Let nrs and m_i be positive integers such that $m_i = r_is_i$ for all $1 \leqslant i \leqslant n$. Then there exists an ODC of $K_{\prod_{i=1}^n m_i,\, \prod_{i=1}^n m_i}$ by $K_{\prod_{i=1}^n r_i,\, \prod_{i=1}^n s_i}$.*

Proof. According to the symmetric starter vector **III**, $v(K_{r_i,s_i})$ are symmetric starter vectors with respect to $\mathbb{Z}_{m_i}$. Applying Theorem (1.4) proves

$$\overbrace{v(K_{r_1,s_1}) \times v(K_{r_2,s_2}) \times \cdots \times v(K_{r_n,s_n})}^{n\ \text{times}} = v\left(K_{\prod_{i=1}^n r_i,\prod_{i=1}^n s_i}\right)$$

is a symmetric starter vector with respect to $\prod_{i=1}^n \mathbb{Z}_{m_i}$. Moreover, the resulting symmetric starter graph has the following edges set:

$$E\left(K_{\prod_{i=1}^n r_i,\prod_{i=1}^n s_i}\right) = \left\{u_0v_1 : \forall u \in \prod_{i=1}^n A_i,\ \ v \in \prod_{i=1}^n \mathbb{Z}_{s_i}\right\}$$

with $A_i = \{x_i{:}x_i = s_i(r_i - j), 1 \leqslant j \leqslant r_i\}$. $\square$

Example 2.1. Let $m_1 = 6$ and $m_2 = 4$, according to the symmetric starter vector **III**,

(1) (0, 0, 0, 3, 3, 3) is a symmetric starter vector of an ODC of $K_{6,6}$ by $K_{2,3}$ with respect to $\mathbb{Z}_6$ and
(2) (0, 0, 2, 2) is a symmetric starter vector of an ODC of $K_{4,4}$ by $K_{2,2}$ with respect to $\mathbb{Z}_4$.

By Theorem (1.4), $v(K_{2,3}) \times v(K_{2,2}) = (00, 00, 02, 02, 00, 00, 02, 02, 00, 00, 02, 02, 30, 30, 32, 32, 30, 30, 32, 32, 30, 30, 32, 32) = v(K_{4,6})$ is a symmetric starter vector of an ODC of $K_{24,24}$ with respect to $\mathbb{Z}_6 \times \mathbb{Z}_4$. Furthermore, edges set of $K_{4,6}$ can be constructed as follows, w.l.o.g. $m_1 = r_1s_1 = 6 \Rightarrow r_1 = 2$, $s_1 = 3$ and hence, $A_1 = \{3(2-1), 3(2-2)\} = \{3, 0\}$. Similar-