Ain Shams Engineering Journal (2013) 4, 515–521

Ain Shams University

Ain Shams Engineering Journal

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CIVIL ENGINEERING

Stability behavior and free vibration of tapered columns with elastic end restraints using the DQM method

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Received 8 July 2012; revised 24 September 2012; accepted 9 October 2012
Available online 11 January 2013

KEYWORDS
Tapered column;
End restraints;
Differential quadrature;
Critical load and natural frequencies

Abstract The stability behavior and free vibration of axially loaded tapered columns with rotational and/or translational end restraints are studied using the differential quadrature method (DQM). The governing differential equations are derived and transformed into a homogeneous system of algebraic equations using the DQM technique. The boundary conditions are discretized and substituted into the governing differential equations, then the problem is transformed into a two parameter eigenvalue problem, namely the critical load and the natural frequency. The solution of the eigenvalue problem yields the critical load for the static case ($\omega = 0$) and yields the natural frequencies for the dynamic case with a prescribed value of axial load ($P_o < P_{cr}$). The obtained solutions were verified against those obtained from FEM and found in close agreement.

1. Introduction

Many practical engineering applications are very sensitive to the weight of the structural elements for different reasons. In space shuttles, the weight of different structural element is optimized as functional requirements, while in ordinary structures the weight optimization is needed for architectural and/or economical issues. To optimize the weight of structural elements in such applications, elements with non-prismatic configurations are commonly used. It is very difficult to obtain closed form solutions representing the behavior of such non-uniform elements under the effect of static and dynamic loads. Often, to obtain closed-form solutions, many idealizations are introduced to simplify the mathematical treatments yielding mathematical models that misrepresent physical models. In addition, the conventional idealization of the end conditions (fixed–hinged–clamped–free) may not represent most situations where support movements are expected and need to be considered in the analysis. Near optimum configurations are studied by many researchers to obtain both stability and vibration behavior of structural elements. Analytical solutions for simple cases of prismatic and non-prismatic elements with elastic end restraints are found in literature [1–3].

Taha and Abohadima [4] studied the vibration of non-uniform shear beam resting on elastic foundation. Semi-analytical methods such as series solutions are suggested to obtain analytic expressions for frequencies and mode shapes of

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Peer review under responsibility of Ain Shams University.

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http://dx.doi.org/10.1016/j.asej.2012.10.005

non-uniform beams resting on elastic foundation [5]. Numerical methods such as the perturbation method [6], Ritz method [7], the finite element method [8–10] and the differential quadrature method [11,12] are used to study certain configurations of such models.

On the other hand, the great expansion in the power of the personal computers and availability of solving algorithms add many advantages to numerical techniques. The differential quadrature method (DQM) is a very efficient numerical method with simple straightforward formulation that needs very limited memory storage and computational time to obtain results with accuracy fair to practical applications.

In the present work, the stability and vibration behavior of axially-loaded tapered columns with translational and/or rotational elastic end restraints are studied using the DQM. The addressed problem was previously solved using the FEM for free vibration of beams (no axial load) [9]. The main differences between the present work and the previous one are the method of solution, the implementation of translational elastic end restraints as well as the presence of the axial load into present analysis. The boundary conditions are discretized and substituted into the governing differential equations. The obtained results are verified against the FEM results and are found to be in close agreement. The effects of numerous tapered configurations for different boundary conditions on the load and frequency parameters are investigated.

2. Formulation of the problem

2.1. Vibration equation

The free vibration equation of a non-prismatic column loaded by an axial force P_o, shown in Fig. 1, is given as:

$$\frac{\partial^2}{\partial X^2}\left[EI(X)\frac{\partial^2 Y}{\partial X^2}\right] + P_o\frac{\partial^2 Y}{\partial X^2} + \rho A(X)\frac{\partial^2 Y}{\partial t^2} = 0. \tag{1}$$

where $I(X)$ is the moment of inertia of the column cross section at X, ρ is the mass density per unit volume, E is the modulus of elasticity, $A(X)$ is the area of cross section at X, $Y(X,t)$ is the lateral displacement, P_o is the axial load acting on the column, X is the distance along the column and t is time.

Using dimensionless parameters $x = X/L$ and $y = Y/L$, Eq. (1) is transformed to:

$$\frac{\partial^2}{\partial x^2}\left[\frac{EI(x)}{L^3}\frac{\partial^2 y}{\partial x^2}\right] + \frac{P_o}{L}\frac{\partial^2 y}{\partial x^2} + \rho A(x)\frac{\partial^2 y}{\partial t^2} = 0. \tag{2}$$

The solution of linear partial differential Eq. (2) is obtained by employing both the separation of variables and the differential quadrature methods. The first step consists in representing the distribution of the lateral displacement by two independent functions, one describing the spatial variation (mode shape function) and the other represents the time variation. The second step consists in using the DQM method to transform the governing differential equation into a homogeneous system of N algebraic equations solved numerically according to the boundary conditions.

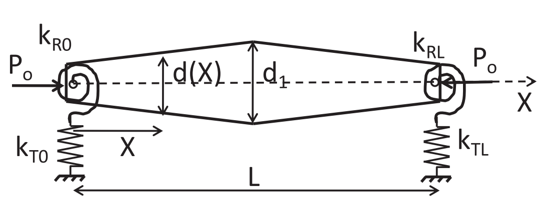

Figure 1 Axially-loaded tapered column with elastic end restraints.

2.2. Boundary conditions

The dimensionless elastic end restraints at $x = 0$ are related to the derivatives of lateral displacement at the column ends as:

$$k_{T0}y(0,t) = -\frac{\partial}{\partial x}\left(\frac{EI_o}{L^3}\frac{\partial^2 y(0,t)}{\partial x^2}\right), \tag{3a}$$

$$k_{R0}\frac{\partial y(0,t)}{\partial x} = \frac{EI_o}{L}\frac{\partial^2 y(0,t)}{\partial x^2}. \tag{3b}$$

Also, the dimensionless elastic end restraints at $x = 1$ are expressed as:

$$k_{TL}y(1,t) = \frac{\partial}{\partial x}\left(\frac{EI_o}{L^3}\frac{\partial^2 y(1,t)}{\partial x^2}\right), \tag{3c}$$

$$k_{RL}\frac{\partial y(1,t)}{\partial x} = -\frac{EI_o}{L}\frac{\partial^2 y(1,t)}{\partial x^2}. \tag{3d}$$

where k_{T0} and k_{TL} are the lateral elastic stiffnesses at $x = 0, 1$ respectively, I_o is the moment of inertia of the column cross section at $x = 0$ and k_{R0} and k_{RL} are the rotational elastic stiffnesses at $x = 0, 1$ respectively.

Following the separation of variables analogy, the solution of Eq. (2) may be assumed as:

$$y(x,t) = y_o\phi(x)\psi(t), \tag{4}$$

where $\phi(x)$ is the linear mode function, $\psi(t)$ is a function representing the time variation and y_o is the dimensionless vibration amplitude (obtained from the initial conditions). Substituting Eq. (4) into Eq. (2), Eq. (2) is separated into:

$$\frac{d^2}{dx^2}\left[\frac{EI(x)}{L^3}\frac{d^2\phi}{dx^2}\right] + \frac{P_o}{L}\frac{d^2\phi}{dx^2} + \rho A(x)\omega^2\phi = 0, \tag{5}$$

$$\frac{d^2\psi}{dt^2} + \omega^2\psi(t) = 0, \tag{6}$$

where ω is the separation constant which represents the natural frequency.

The solution of Eq. (6), assuming at $t = 0$, $\psi = 1.0$ and $d\psi/dt = 0$ is:

$$\psi(t) = \cos(\omega t). \tag{7}$$

The general solution of Eq. (5) depends on the distribution of the section geometry along the column. Fig. 1 shows the case of a symmetric tapered column, where the depth of the column increases lineally from d_o at $x = 0$ to d_1 at $x = 0.5$, then decreases linearly form d_1 at $x = 0.5$ to d_o at $x = 1$, while the width of the column b is assumed constant, then:

$$d(x) = d_o\eta(x), \tag{8}$$

where

$$\eta(x) = \begin{cases} 1 - 2x(1-\alpha) & \text{for } 0.0 \leqslant x \leqslant 0.5 \\ 2\alpha + 2x(1-\alpha) - 1 & \text{for } 0.5 \leqslant x \leqslant 1.0 \end{cases}$$

and $\alpha = d_1/d_o$ is the tapering ratio.

Using the distribution of section geometry expressed in Eq. (8), the distribution of area and moment of inertia of the column cross section are given as: