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Analytical solutions of time-fractional models for homogeneous Gardner equation and non-homogeneous differential equations

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Abstract In this paper, we obtain analytical solutions of homogeneous time-fractional Gardner equation and non-homogeneous time-fractional models (including Buck-master equation) using q-Homotopy Analysis Method (q-HAM). Our work displays the elegant nature of the application of q-HAM not only to solve homogeneous non-linear fractional differential equations but also to solve the non-homogeneous fractional differential equations. The presence of the auxiliary parameter h helps in an effective way to obtain better approximation comparable to exact solutions. The fraction-factor in this method gives it an edge over other existing analytical methods for non-linear differential equations. Comparisons are made upon the existence of exact solutions to these models. The analysis shows that our analytical solutions converge very rapidly to the exact solutions.

1. Introduction

The frequently used analytical methods to solve non-linear differential equations have different restrictions and discretization of variables are involved in numerical techniques which leads to rounding off errors see [19].

The Gardner equation (combined KdV–mKdV or eKdV equation) is a useful model for the description of internal solitary waves in shallow water while the buck-master's equation is used in thin viscous fluid sheet flows and have been widely studied by the various methods see [2,21].

Generally, for the past three decades, fractional calculus has been considered with great importance due to its various applications in physics, fluid flow, chemical physics, control theory of dynamical systems, electrical networks, and so on. The quest of getting accurate methods for solving resulted non-linear models involving fractional order is of utmost concern of many researchers in this field today.

Various analytical methods have been put to use successfully to obtain solutions of classical Gardner equations and Buck-Master equations such as the method of planar dynamical systems approach, exp-function method, bilinear method and extended homo-clinic test approach, fractional variational iteration method (FVIM) and generalized double reduction theorem see [1,3–7,13,16–18]. Recently, a modified HAM called q-Homotopy Analysis Method was introduced in [8], see also [10–12]. It was proven that the presence of fraction

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factor in this method enables a fast convergence better than the usual HAM which then makes is more reliable.

To the best of our knowledge, no attempt has been made regarding analytical solutions of time-fractional homogeneous Gardner equation and time fractional non-homogeneous Buck-Master equation using q-Homotopy Analysis Method. In this paper, we consider these equations subject to some appropriate initial conditions. Comparison analysis of our results is carried out with exact solutions when they exist. The numerical results of the problems are presented graphically, obtained using Mathematica 9 and MATLAB R2012b.

2. Preliminaries

This section is devoted to some definitions and some known results. Caputo's fractional derivative is adopted in this work.

Definition 2.1. The Riemann–Liouville's (RL) fractional integral operator of order $\alpha \geqslant 0$, of a function $f \in L^1(a,b)$ is given as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \quad t>0, \quad \alpha>0, \tag{1}$$

where Γ is the Gamma function and $I^0 f(t) = f(t)$.

Definition 2.2. The fractional derivative in the Caputo's sense is defined as [20],

$$\mathcal{D}^\alpha f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \tag{2}$$

where $n-1 < \alpha \leqslant n, n \in \mathbb{N}, t > 0$.

Lemma 2.1. *Let $t \in (a,b]$. Then*

$$\left[I_a^\alpha (t-a)^\beta\right](t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} (t-a)^{\beta+\alpha}, \qquad \alpha \geqslant 0, \quad \beta > 0. \tag{3}$$

Definition 2.3. The Mittag-Leffler function for two parameters is defined as,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad \alpha, \beta, z \in \mathcal{C}, \quad Re(\alpha) \geqslant 0 \tag{4}$$

3. q-Homotopy Analysis Method (q-HAM)

Differential equation of the form

$$N\left[\mathcal{D}_t^\alpha u(x,t)\right] - f(x,t) = 0 \tag{5}$$

is considered, where N is a nonlinear operator, $\mathcal{D}_t^\alpha$ denotes the Caputo fractional derivative, (x,t) are independent variables, f is a known function and u is an unknown function. To generalize the original Homotopy method, Liao [9] construct what is generally known as the zeroth-order deformation equation

$$\begin{aligned}(1-nq)L(\phi(x,t;q) - u_0(x,t)) \\ = qhH(x,t)\left(N[\mathcal{D}_t^\alpha \phi(x,t;q)] - f(x,t)\right),\end{aligned} \tag{6}$$

where $n \geqslant 1, q \in \left[0, \frac{1}{n}\right]$ denotes the so-called embedded parameter, L ia an auxiliary linear operator, $h \neq 0$ is an auxiliary parameter, $H(x,t)$ is a non-zero auxiliary function.

It is clearly seen that when $q = 0$ and $q = \frac{1}{n}$, Eq. (1) becomes

$$\phi(x,t;0) = u_0(x,t) \quad \text{and} \quad \phi\left(x,t;\frac{1}{n}\right) = u(x,t) \tag{7}$$

respectively. So, as q increases from 0 to $\frac{1}{n}$, the solution $\phi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$.

If $u_0(x,t), L, h, H(x,t)$ are chosen appropriately, solution $\phi(x,t;q)$ of Eq. (1) exists for $q \in \left[0, \frac{1}{n}\right]$.

Expansion of $\phi(x,t;q)$ in Taylor series gives

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m. \tag{8}$$

where

$$u_m(x,t) = \frac{1}{m!} \frac{\partial^m \phi(x,t;q)}{\partial q^m}\bigg|_{q=0}. \tag{9}$$

Assume that the auxiliary linear operator L, the initial guess u_0, the auxiliary parameter h and $H(x,t)$ are properly chosen such that the series (8) converges at $q = \frac{1}{n}$, then we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) \left(\frac{1}{n}\right)^m. \tag{10}$$

Let the vector u_n be define as follows:

$$\vec{u}_n = \{u_0(x,t), u_1(x,t), \ldots, u_n(x,t)\}. \tag{11}$$

Differentiating Eq. (6) m-times with respect to the (embedding) parameter q, then evaluating at $q = 0$ and finally dividing them by $m!$, we have what is known as the mth-order deformation equation (Liao [14,15]) as

$$L\left[u_m(x,t) - \chi_m^* u_{m-1}(x,t)\right] = hH(x,t)\mathcal{R}_m(\vec{u}_{m-1}). \tag{12}$$

with initial conditions

$$u_m^{(k)}(x,0) = 0, \quad k = 0,1,2,\ldots,m-1. \tag{13}$$

where

$$\mathcal{R}_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1}\left(N[\mathcal{D}_t^\alpha \phi(x,t;q)] - f(x,t)\right)}{\partial q^{m-1}}\bigg|_{q=0} \tag{14}$$

and

$$\chi_m^* = \begin{cases} 0 & m \leqslant 1 \\ n & \text{otherwise,} \end{cases} \tag{15}$$

Remark 1. It should be emphasized that $u_m(x,t)$ for $m \geqslant 1$, is governed by the linear operator (12) with the linear boundary conditions that come from the original problem. The existence of the factor $\left(\frac{1}{n}\right)^m$ gives more chances for better convergence, faster than the solution obtained by the standard Homotopy method. Off course, when $n = 1$, we are in the case of the standard Homotopy method.

4. The time-fractional Garner equation

We consider the time fractional homogeneous time-fractional Garner equation. Let

$$\begin{aligned}&D_t^\alpha u + 6(u - \varepsilon^2 u^2)u_x + u_{xxx} = 0, \qquad 0 \leqslant x \leqslant 1, \quad t > 0, \\ &0 < \alpha \leqslant 1\end{aligned} \tag{16}$$

subjected to the initial condition

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