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New techniques for solving some matrix and matrix differential equations

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Abstract Matrix and matrix differential equations play an important role in system theory, control theory, stability theory of differential equations, communication systems and many other fields. In this paper, we present the solutions of non-homogeneous matrix differential equations, convolution matrix differential equations and matrix equations which include the renewal matrix equation by using convolution and Kronecker products of matrices. Furthermore, the existence and uniqueness of the solution of some important and interesting special cases of these equations are also considered with some illustrated examples in order to show our new approaches.

1. Introduction and preliminary results

In addition to the matrix usual multiplication; there has been renewed interest in two kinds of matrix multiplication. These multiplications are the convolution and Kronecker products which are playing very important roles in many applications and the technique has successfully applied in various fields of pure and applied mathematics, for example, in the solution of matrix and matrix differential equations [1–15]. The notations: $M_{m,n}$, A^T, A^{-1}, A^+, $rank(A)$, e^A, $||A||$, $\sigma(A)$ are stand to the set of all $m \times n$ matrices (when $m = n$, we write M_n instead of $M_{n,n}$), transpose, inverse, Moore–Penrose inverse, rank, exponential, norm, and the set of all eigenvalues of a matrix A, respectively. Now, we recall the main definitions and some important properties of the Kronecker and convolution products of matrices that will be very useful in our investigation in the solution of matrix equations and matrix differential equations.

The Kronecker and convolution products used in many fields are almost as important as the usual product. One of the principle reasons is that these products affirming their capability of solving a wide range of problems and playing important tools in many fields such as control theory, system theory, statistics, physics, communication systems, optimization, economics and engineering. These include signal processing, image processing, semi definite programming, matrix equations, matrix differential equations and many other applications [1–21]. The following four matrix operations are studied by many researchers [1–4,7–14,20,21] and defined as follow:

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(i) *Kronecker product:*

$$A \otimes B = (a_{ij}B)_{ij} \in M_{mp,nq}, \quad (1\text{-}1)$$

where $A = (a_{ij}) \in M_{m,n}$ and $B = (b_{kl}) \in M_{p,q}$.

(ii) *Kronecker sum:*

$$A \oplus B = (A \otimes I_n) + (I_m \otimes B) \in M_{mn}, \quad (1\text{-}2)$$

where $A = (a_{ij}) \in M_m$ and $B = (b_{kl}) \in M_n$.

(iii) *Vector operator:*

$$VecA = (a_{11}a_{21}\ldots a_{m1}a_{12}a_{22}\ldots a_{m2}\ldots a_{1n}a_{2n}\ldots a_{mn})^T \in M_{mn,1}, \quad (1\text{-}3)$$

where $A = (a_{ij}) \in M_{m,n}$.

(iv) *Convolution product:*

$$A * B(t) = (h_{ir}(t)) \text{ with } h_{ir}(t) = \sum_{k=1}^{n}\int_0^t f_{ik}(t-x)g_{kr}(x)dx = \sum_{k=1}^{n} f_{ik} * g_{kr}(t), \quad (1\text{-}4)$$

where $A(t) = (f_{ij}(t)) \in M_{m,n}$ and $B(t) = (g_{jr}(t)) \in M_{n,p}$ are integrable matrices for all $t \geqslant 0$, such that $f_{ij}(t)$ and $g_{jr}(t)$ are well-defined functions for all positive integer values i, j, r.

The following three definitions are also very useful in our investigation in the solutions of renewal matrix equations and matrix differential convolution equations. If $A(t) = (f_{ij}(t)) \in M_n$ is an integrable matrix, then [5–6,20]

(i) The *m-power matrix convolution product* of $A(t)$ is defined by

$$A^{\{m\}}(t) = A * A * \cdots * A(t) = (f_{ij}^{\{m\}}(t)) \in M_n \text{ with } f_{ij}^{\{m\}}(t) = \sum_{k=1}^{n} f_{ik}^{\{m-1\}} * f_{kj}(t), \quad (1\text{-}5)$$

where m is positive integer number, and $A^{\{1\}}(t) = A(t)$.

(ii) The determinant of $A(t)$ is defined by

$$\det A(t) = \sum_{j=1}^{n}(-1)^{j+1} f_{1j} * D_{1j}, \quad (1\text{-}6)$$

where D_{ij} is the determinant of the $(n-1)\times(n-1)$ matrix function obtained from $A(t)$ by deleting row i and column j of $A(t)$. We call D_{ij} the minor of $A(t)$ corresponding to the entry $f_{ij}(t)$ of $A(t)$.

(iii) If $\det(A(t)) \neq 0$, the inversion of $A(t)$ is defined by

$$A^{\{-1\}}(t) = (h_{ij}(t)) \text{ with } h_{ij}(t) = [\det(A(t))]^{\{-1\}} * adjA(t). \quad (1\text{-}7)$$

For any compatibly matrices A, B, C and D, we shall make frequent use of the following properties of the Kronecker products [1–4,7–14].

(i) $(A \otimes B)^T = A^T \otimes B^T$ (1-8)

(ii) $(A \otimes B)^+ = A^+ \otimes B^+$ (1-9)

(ii) $rank(A \otimes B) = rank(A)rank(B)$ (1-10)

(iii) $e^{(A\oplus B)} = e^A \otimes e^B$ (1-11)

(iv) $(A \otimes B)(C \otimes D) = AC \otimes BD$ (1-12)

(v) $I_m \otimes I_n = I_n \otimes I_m = I_{mn}$, where I_m is the identity *matrix of order* $m \times m$. (1-13)

(vi) If $\sigma(A) = \{\lambda_i : i = 1, 2, \ldots, m\}$ and $\sigma(B) = \{\mu_j : j = 1, 2, \ldots, n\}$ are the set of eigenvalues of $A \in M_m$ and $B \in M_n$, respectively. Then

(i) $\sigma(A \otimes B) = \{\lambda_i\mu_j : i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$. (1-14)

(ii) $\sigma(A \oplus B) = \{\lambda_i + \mu_j : i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$. (1-15)

(vii) $f(A \otimes I_n) = f(A) \otimes I_n, f(I_n \otimes A) = I_n \otimes f(A)$, (1-16)

where f is analytic function on the region containing the eigenvalues of $A \in M_m$ such that $f(A)$ exist.

Some special cases include (1-16):

(i) $e^{A\otimes I} = e^A \otimes I$ and $e^{I\otimes A} = I \otimes e^A$ (1-17)

(ii) $\sinh(A \otimes I) = \sinh(A) \otimes I$ and $\sinh(I \otimes A) = I \otimes \sinh(A)$ (1-18)

(iii) $\cosh(A \otimes I) = \cosh(A) \otimes I$ and $\cosh(I \otimes A) = I \otimes \cosh(A)$. (1-19)

For any matrix $A \in M_m$, the spectral representation of e^A and e^{At} assures that:

$$e^A = \sum_{i=0}^{n} x_i y_i^T e^{\lambda_i}, e^{At} = \sum_{i=0}^{n} x_i y_i^T e^{\lambda_i t}, \quad (1\text{-}20)$$

where $\{\lambda_1, \cdots, \lambda_n\}$ are the eigenvalues of A, $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ are the set of all eigenvectors of A and A^T, respectively, corresponding to the eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$.

The nice relationship between the Kronecker product and vector-operator is given by [1–4,7–14]

$$Vec(AXB) = (B^T \otimes A)VecX, \quad (1\text{-}21)$$

where $A \in M_{m,n}$, $B \in M_{p,q}$ and $X \in M_{n,p}$.

For any compatibly integrable matrices $A(t) = (f_{ij}(t))$ and $B(t) = (g_{ij}(t))$, we shall make frequent use of the following properties of the convolution product [18,20,21]:

(i) $\pounds(A * B) = \pounds((A))(s)\pounds((B))(s)$ (1-22)

(ii) $(A * B(t))(i, r) \leqslant (A(t)B(t))(i, r)$ (t fixed) (1-23)

(iii) $\|A * B(t)\| \leqslant \|A(t)\|.\|B(t)\|$ (1-24)

(iv) $\|A^{\{m\}}(t)\| \leqslant \|A(t)\|^m$ (m is positive integer) (1-25)

Finally, the Moore–Penrose inverse is widely used in perturbation theory, singular systems, neural network problems, least-squares problems, optimization problems and many other subjects. The Moore–Penrose inverse of an arbitrary matrix $A \in M_{m,n}$ is defined to be the unique solution of the following four matrix equations [4,11,14]:

$$AXA = A,\ XAX = X,\ (AX)^T = AX,\ (XA)^T = XA, \quad (1\text{-}26)$$

and is often denoted by $X = A^+ \in M_{n,m}$. Note that if $A \in M_{m,n}$, then we have the following special cases:

(i) If $rank(A) = m = n$, then $A^+ = A^{-1}$ (1-27)

(ii) If $rank(A) = n, A^+ = (A^TA)^{-1}A^T$ and $A^+A = I_n$ (1-28)

(iii) If $rank(A) = m, A^+ = A^T(AA^T)^{-1}$ and $AA^+ = I_m$. (1-29)

In the present paper, based on the vector-operator, Kronecker products and convolution products of matrices, we present the general solutions of some matrix and matrix differential equations. These equations involve the renewal matrix equation, general matrix equation and non-homogeneous matrix differential equations, then we show that the