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Optimal harvesting strategy and stochastic analysis for a two species commensaling system

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Abstract In this paper, we have considered a mathematical model of commensalism between two species (S_1 and S_2) with a limited resource of food, in addition the paper also highlights how the commensal and host species are harvested. The model is characterized by a couple of first order non-linear differential equations. Here, the stable equilibrium point is identified and its stability (both local and global) criteria are discussed (both analytical and numerical). An optimal harvesting strategy is being conversed using Pontriyagin's maximum principle. We have explored the stochastic stability by finding the corresponding variances. Finally numerical simulations illustrate the effectiveness of our results.

1. Introduction

Ecology is the study of relationships between living organisms and their environment. Research in the area of theoretical ecology was started by Lotka [10] and Volterra [19]. Since then many mathematicians and ecologists have contributed to the growth of this area creating awareness as reported in the dissertations of Meyer [11], Cushing [4], Paul Colinvaux [14], Kapur [5,6], etc. The ecological interactions can be extensively classified as ammensalism, neutralism, commensalism, competition, predation, and so forth. Srinivas [17] deliberated competitive eco-system of two species and three species with limited and unlimited resources. Later, Lakshminarayan and Pattabhiramacharyulu [8,9] premeditated prey predator ecological models with a partial cover for the prey and alternate food for the predator. In recent times stability analysis of competitive species was carried out by Archana Reddy et al. [1] and Sharma and Pattabhiramacharyulu [2], whereas Ravindra Reddy [16] investigated mutualism between two species. In 1996, Mesterton-Gibbons [12] described the skills to find the finest harvesting strategy for a Lotka–Volterra eco-system of two independent inhabitants. He also advocated that the technique may be extensively applicable in ecological modeling and other recent claims. In 2009, Phanikumar et al. [15] inspected the stability conditions for a mathematical model of commensalism between two species S_1 and S_2 with limited resources; the linearized disturbed equations are solved and the trajectories are illustrated. In 2005, Kar and Swarnakamal [7] proposed a prey predator model in a two patch environment: 1.

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Nomenclature

x	biomass density of commensal species	a_{ii}, $i = 1, 2$	of decrease in S_i due to limitations of natural resources
y	biomass density of host species	a_{12}	commensal coefficient
S_1	commensal species	q_1	represents the catchability coefficient of S_1 species
S_2	host species	E_1	effort applied to harvest the S_1 species
a_i, $i = 1, 2$	natural growth rates of S_i		

Accessible to both prey and predators (patch 1) and 2. Being a refuge for the prey (patch 2). They assumed that the prey refuge (patch 2) constitutes a reserve zone of prey and fishing is not permitted, while the unreserved area is an open-access fishery zone. The existence of possible steady state points along with their local and global stability is discussed. They also examined the possibilities of the existence of bionomic equilibrium. Phanikumar et al. [15], Kar and Swarnakamal [7], and Carletti [3] inspired us to consider a commensalism model, incorporating harvesting in commensal species with a stochastic term. The present exploration is devoted to the analytical and numerical comparisons of commensalism with harvesting for commensal species. This also includes stochastic immovability. Two species commensalism is an ecological relationship between two species where one species S_1 derives benefit from the other species S_1 which would not get affected by it. S_1 may be referred as the commensal species, while S_1 is the host. Some of the examples are cattle Egrat, Anemonetish, Barnacles, etc. The host species S_1 supports the commensal species S_1 and has its own natural growth rate in spite of a support apart from S_2. The commensal species S_1 in spite of the limitation of its natural resources flourishes drawing strength from the host species S_2. The model is characterized by a couple of first order non-linear differential equations. All the four steady state points of the system are recognized and their stability analysis is carried out. It is detected that the co-existence state is the only stable state that pertains to specified clauses. However, the other three steady states are unstable.

2. Basic mathematical model

$$(dx)/(dt) = x[(a_1 - q_1E_1) - a_{11}x + a_{12}y] \tag{2.1}$$

$$(dy)/(dt) = y[a_2 - a_{22}y] \tag{2.2}$$

where $x(t)$ represents the biomass density of commensal species S_1, $y(t)$ represents the biomass density of host species S_2. a_i, $i = 1, 2$ represents the natural growth rates of S_i. a_{ii}, $i = 1, 2$ represents the rate of decrease in S_i due to limitations of natural resources. a_{12} represents the commensal coefficient. q_1 represents the catchability coefficient of S_1 species. E_1 represents the effort applied to harvest the S_1 species. Throughout our analysis, let us assume that

$$a_1 - q_1E_1 > 0 \tag{2.3}$$

3. Analysis of steady states

The possible equilibrium points are E_1 (0,0), $E_2(\bar{x}, 0)$, $E_3(0, \bar{y})$, and E_4 (x^*, y^*).

Case (i): E_1 (0,0): This equilibrium point always exist.
Case (ii): $E_2(\bar{x}, 0)$:
Here $\bar{x}$, is the positive solution of $(dx)/(dt) = 0$, which gives

$$\bar{x} = [1/(a_{11})](a_1 - q_1E_1) \tag{3.1}$$

Clearly we observe that (3.1) is positive due to inequality (2.3).
Case (iii): $E_3(0, \bar{y})$:
Here $\bar{y}$ is the positive solution of $(dy)/(dt) = 0$, which gives

$$\bar{y} = a_2/(a_{22}) \tag{3.2}$$

Case (iv): E_4 (x^*, y^*) (The interior equilibrium):
Here x^* and y^* are positive solutions of $(dx)/(dt) = 0$ and $(dy)/(dt) = 0$, which gives

$$y^* = a_2/(a_{22}) \tag{3.3}$$

$$x^* = [1/(a_{11})][(a_1 - q_1E_1) + [(a_2a_{12})/a_{22}]] \tag{3.4}$$

Clearly we have identified that (3.4) is positive due to the inequality (2.3).

4. Local stability

To determine the local stability character of the interior equilibrium $E_4(x^*, y^*)$, we compute the variational matrix about E_4.

$$J(x, y) = \begin{bmatrix} a_1 - 2a_{11}x + a_{12}y + a_{13}z - q_1E_1 & a_{12}x \\ 0 & a_2 - 2a_{22}y \end{bmatrix} \tag{4.1}$$

The characteristic equation of (4.1) at the interior equilibrium E_4 (x^*, y^*) is

$$(a_{11}x^* + \lambda)(a_{22}y^* + \lambda) = 0 \tag{4.2}$$

The roots $\lambda_1 = -a_{11}x^*$; $\lambda_2 = -a_{22}y^*$ of the Eq. (4.2) are both negative. Hence the steady state is stable. Since $\lambda_1 + \lambda_2 = -(a_{11}x^* + a_{22}y^*) < 0$ and $\lambda_1\lambda_2 = -a_{11}a_{22}x^*y^* > 0$, E_4 (x^*, y^*) is locally asymptotically stable.

5. Global stability

Theorem: The equilibrium point E_4 (x^*, y^*) is globally asymptotically stable.

Proof: Let us consider the following Lyapunov function

$$V(x, y) = [(x - x^*) - x^* \ln(x/x^*)] + l_1[(y - y^*) - y^* \ln(y/y^*)]$$

where l_1 is the positive constant.
$(dV)/(dt) = [(x - x^*)/x][dx/dt] + l_1[(y - y^*)/y][dy/dt]$;