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(1 + *n*)-Dimensional Burgers' equation and its analytical solution: A comparative study of HPM, ADM and DTM

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Abstract In this article, we present homotopy perturbation method, adomian decomposition method and differential transform method to obtain a closed form solution of the (1 + *n*)-dimensional Burgers' equation. These methods consider the use of the initial or boundary conditions and find the solution without any discritization, transformation, or restrictive conditions and avoid the round-off errors. Four numerical examples are provided to validate the reliability and efficiency of the three methods.

1. Introduction

Consider the (1 + *n*)-dimensional Burgers' equation

$$\frac{\partial u}{\partial t} = \alpha_1 \frac{\partial^2 u}{\partial x_1^2} + \alpha_2 \frac{\partial^2 u}{\partial x_2^2} + \alpha_3 \frac{\partial^2 u}{\partial x_3^2} + \cdots + \alpha_n \frac{\partial^2 u}{\partial x_n^2} + \beta u \frac{\partial u}{\partial x_1}, \tag{1}$$

under the initial condition

$$u(x_1, x_2, x_3, \ldots, x_n, 0) = u_0(x_1, x_2, x_3, \ldots, x_n), \tag{2}$$

where α_i, $i = 1, 2, 3, \ldots, n$, and β are constants.

Eq. (1) is also known as Richard's equation, which is used in the study of cellular automata, and interacting particle systems. Eq. (1) describes the flow pattern of the particle in a lattice fluid past an impenetrable obstacle [1,2]; it can be also used as a model to describe the water flow in soils.

One and two dimensional Burgers' equations are quite famous in wave theory, which has applications in gas dynamics [3] and in plasma physics. Due to its broad range of applications, various studies have been made to generalize it to higher dimensions.

In the past few decades, traditional integral transform methods such as Fourier and Laplace transforms have commonly been used to solve engineering problems. These methods transform differential equations into algebraic equations which are easier to deal with. However, these integral transform methods are more complex and difficult when applying to nonlinear problems. Homotopy perturbation method (HPM) is a new analytical method which was firstly proposed

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by He [4–6] to solve linear and nonlinear differential equations. The method is a coupling of the traditional perturbation method and homotopy in topology. HPM deforms a difficult problem into a simple problem which can be easily solved. It is the most effective and convenient method for both linear and nonlinear equations. Unlike the traditional numerical methods, homotopy perturbation method does not need linearization, discretization, or any other transformation; large computational work and round-off errors are avoided. It has fast convergence and high accuracy. Most perturbation methods assume a small parameter exists, but most of the nonlinear problems do not have small parameter at all. Many new methods have been proposed to eliminate the small parameter. Recently, the applications of homotopy theory among researchers and scientists appeared, and the homotopy theory becomes a powerful mathematical tool, when it is successfully coupled with perturbation theory.

Adomian decomposition method (ADM), which was introduced by Adomian [7], is a semi-numerical technique for solving linear and nonlinear differential equations by generating a functional series solution in a very efficient manner. The method has many advantages: it solves the problem directly without the need for linearization, perturbation, or any other transformation; it converges very rapidly and is highly accurate.

Differential transform method (DTM), which was first applied in the engineering field by Zhou [8], has many advantages: it solves the problem directly without the need for linearization, perturbation, or any other transformation. DTM is based on the Taylor's series expansion. It constructs an analytical solution in the form of a polynomial. It is different from the traditional high order Taylor series method, which needs symbolic computation of the necessary derivatives of the data functions. Taylor series method computationally takes a long time for larger orders while DTM method reduces the size of the computational domain, without massive computations and restrictive assumptions, and is easily applicable to various physical problems. The method and related theorems are well addressed in [9,10].

In this work, we have employed HPM, ADM and DTM to solve $(1 + n)$-dimensional Burgers' equation. Four numerical examples are carried out to validate and illustrate the above three methods.

2. Homotopy Perturbation Method (HPM)

To describe HPM, consider the following general nonlinear differential equation

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{3}$$

under the boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \partial\Omega, \tag{4}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, and $\partial\Omega$ is boundary of the domain Ω. The operator A can be divided into two parts L and N, where L is a linear operator while N is a nonlinear operator. Then Eq. (3) can be rewritten as

$$L(u) + N(u) - f(r) = 0, \tag{5}$$

Using the homotopy technique, we construct a homotopy: $v(r,p) : \Omega \times [0,1] \to \mathbb{R}$ which satisfies

$$\left.\begin{array}{l} H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)], \\ or \\ H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)], \end{array}\right\}, \quad r \in \Omega, \quad p \in [0,1], \tag{6}$$

where $p \in [0,\ 1]$ is an embedding parameter, u_0 is the initial approximation of Eq. (3) which satisfies the boundary conditions. Obviously, considering Eq. (6), we will have

$$\left.\begin{array}{l} H(v,0) = L(v) - L(u_0) = 0, \\ H(v,1) = A(v) - f(r) = 0, \end{array}\right\}. \tag{7}$$

Changing process of p from zero to unity is just that $v(r,\ p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(v)$ and $A(v) - f(r)$ are called homotopy. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter [4–6] to obtain

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \cdots = \sum_{n=0}^{\infty} p^n v_n. \tag{8}$$

$p \to 1$ results the approximate solution of Eq. (3) as

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots = \sum_{n=0}^{\infty} v_n. \tag{9}$$

A comparison of like powers of p give the solutions of various orders.

Series (9) is convergent for most of the cases. However, convergence rate depends on the nonlinear operator, $N(v)$.

He [6] suggested the following opinions:

1. The second derivative of $N(v)$ with respect to v must be small as the parameter p may be relatively large.
2. The norm of $L^{-1}\frac{\partial N}{\partial v}$ must be smaller than one so that the series converges.

Now we implement HPM method to Eq. (1). According to HPM, we construct the following simple homotopy

$$\frac{\partial u}{\partial t} + p\left[\left(-\alpha_1 \frac{\partial^2 u}{\partial x_1^2} - \alpha_2 \frac{\partial^2 u}{\partial x_2^2} - \alpha_3 \frac{\partial^2 u}{\partial x_3^2} - \cdots - \alpha_n \frac{\partial^2 u}{\partial x_n^2}\right) - \beta u \frac{\partial u}{\partial x_1}\right] = 0. \tag{10}$$

With the initial approximation $u(x_1,\ x_2,\ x_3,\ \ldots,\ x_n,\ 0) = u_0(x_1,\ x_2,\ x_3,\ \ldots,\ x_n)$, we suppose that the solution has following form

$$\begin{aligned} u(x_1, x_2, x_3, \ldots, x_n, t) &= u_0(x_1, x_2, x_3, \ldots, x_n, t) \\ &\quad + pu_1(x_1, x_2, x_3, \ldots, x_n, t) \\ &\quad + p^2 u_2(x_1, x_2, x_3, \ldots, x_n, t) \\ &\quad + p^3 u_3(x_1, x_2, x_3, \ldots, x_n, t) + \cdots \end{aligned} \tag{11}$$

Now substituting from Eq. (11) into Eq. (10) and equating the terms with like powers of p, we get

$$p^0 : \frac{\partial u_0}{\partial t} = 0, \tag{12}$$

$$p^1 : \frac{\partial u_1}{\partial t} = \left(\alpha_1 \frac{\partial^2 u_0}{\partial x_1^2} + \alpha_1 \frac{\partial^2 u_0}{\partial x_2^2} + \alpha_1 \frac{\partial^2 u_0}{\partial x_3^2} + \cdots + \alpha_1 \frac{\partial^2 u_0}{\partial x_n^2}\right) + \beta u \frac{\partial u_0}{\partial x_1}, \tag{13}$$