Ain Shams Engineering Journal (2012) 3, 417–422

Ain Shams University

Ain Shams Engineering Journal

www.elsevier.com/locate/asej
www.sciencedirect.com

ENGINEERING PHYSICS AND MATHEMATICS

Legendre approximation solution for a class of higher-order Volterra integro-differential equations

S.G. Venkatesh *, S.K. Ayyaswamy, S. Raja Balachandar

Department of Mathematics, School of Humanities and Sciences, SASTRA University, Thanjavur 613 401, Tamil Nadu, India

Received 10 August 2011; revised 13 January 2012; accepted 9 April 2012
Available online 24 August 2012

KEYWORDS

Legendre polynomials;
Legendre wavelets;
Integro-differential equations;
Gaussian integration;
Legendre wavelet method

Abstract The aim of this work is to study the Legendre wavelets for the solution of boundary value problems for a class of higher order Volterra integro-differential equations using function approximation. The properties of Legendre wavelets together with the Gaussian integration method are used to reduce the problem to the solution of nonlinear algebraic equations. Also a reliable approach for convergence of the Legendre wavelet method when applied to a class of nonlinear Volterra equations is discussed. Illustrative examples have been discussed to demonstrate the validity and applicability of the technique and the results obtained by Legendre wavelet method is very nearest to the exact solution. The results demonstrate reliability and efficiency of the proposed method.

1. Introduction

Integro-differential equation (IDE) is an equation that the unknown function appears under the sign of integration and it also contains the derivatives of the unknown function.

Mathematical modeling of real-life problems usually results in functional equations, e.g. partial differential equations, integral and integro-differential equations, stochastic equations and others. Many mathematical formulations of physical phenomena contain integro-differential equations, these equations arise in fluid dynamics, biological models and chemical kinetics. In the past several decades, many effective methods for obtaining approximation/numerical solutions of linear/nonlinear differential equations have been presented, such as Adomian decomposition method [1], variational iteration method [2,3], homotopy perturbation method [4–6], He's homotopy perturbation method [7–10], homotopy analysis method [11], and wavelet methods [12–14,4,5].

The literature of numerical analysis contains little on the solution of the boundary value problems for higher-order integro-differential equations. The boundary value problems for higher-order integro-differential equations had been investigated by Morchalo [16,17] and Agarwal [18] among others. Agarwal [18] discussed the existence and uniqueness of the solutions for these problems. The following gives the details

* Corresponding author. Tel.: +91 4362 264101x108; fax: +91 4362 264120.
E-mail addresses: sgv@maths.sastra.edu (S.G. Venkatesh), sjcayya@yahoo.co.in (S.K. Ayyaswamy), srbala09@gmail.com (S. Raja Balachandar).

Peer review under responsibility of Ain Shams University.

ELSEVIER Production and hosting by Elsevier

2090-4479 © 2012 Ain Shams University. Production and hosting by Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.asej.2012.04.007

of literature survey for the integro-differential equations of lower order.

Ghasemi et al. [10] presented He's homotopy perturbation method for solving nonlinear integro-differential equations. Zhao and Corless [19] adopted finite difference method for integro-differential equations. Yusufoglu (Agadjanov) [20] had solved initial value problem for Fredholm type linear integro-differential equation system. Seyed Alizadeh et al. [21] discussed an Approximation of the analytical solution of the linear and nonlinear integro-differential equations by Homotopy Perturbation Method. Wazwaz [22] gave a reliable algorithm for solving boundary value problems for higher-order integro-differential equations. Lepik [23] had solved the nonlinear integro-differential equations using Haar wavelet method. Ghasemi et al. [4] discussed the comparison between wavelet Galerkin method and homotopy perturbation method for the nonlinear integro-differential equations. Ghasemi et al. [15,5] established numerical solution of linear integro-differential equations by using sine–cosine wavelet method and they have also compared with homotopy perturbation method.

In recent years, wavelets have found their way into many different fields of science and engineering. Many researchers started using various wavelets [12–14,4,5,13] for analyzing problems of greater computational complexity and proved wavelets to be powerful tools to explore new direction in solving differential equations. Legendre wavelet based approximate solution of Lane-Emden type was studied by Yousefi [24] recently.

In the present article, we apply Legendre wavelet method (LWM) to find the approximate solution of mth order integro-differential equation [22] of the form

$$y^{(m)}(x) = f(x) + \int_0^x K(x,t)F(y(t))dt, \quad 0 < x < b \tag{1}$$

With the boundary conditions

$$y^{(j)}(0) = \alpha_j, \quad j = 0,1,2,\ldots,(r-1)$$
$$y^{(j)}(b) = \beta_j, \quad j = r,(r+1),\ldots,(m-1)$$

where $y^{(m)}(x)$ indicates the mth derivative of $y(x)$, and $F(y(x))$ is a nonlinear function. In addition, the kernel $K(x,t)$ and $f(x)$ are given in advance. It is of interest to point out that $y(x)$ and $f(x)$ are assumed real and as many times differentiable as required for $x \in [0,b]$, and α_j, $0 \leqslant j \leqslant (r-1)$ *and* $\beta_j, r \leqslant j \leqslant (m-1)$ are real finite constants.

The Legendre wavelet method (LWM) consists of conversion of integro-differential equations into integral equations and expanding the solution by Legendre wavelets with unknown coefficients. The properties of Legendre wavelets together with the Gaussian integration formula are then utilized to evaluate the unknown coefficients and find an approximate solution to Eq. (1).

The organization of the paper is as follows: In Section 2, we describe the basic formulation of wavelets and Legendre wavelets required for our subsequent development. Section 3 is devoted to the solution of Eq. (1) by using integral operator and Legendre wavelets. Convergence analysis and the error estimation for the proposed method have been discussed in Section 4. In Section 5, we report our numerical finding and demonstrate the accuracy of the proposed scheme by considering numerical examples with error calculations. Concluding remarks are given in the final section.

2. Properties of Legendre wavelets

2.1. Wavelets and Legendre wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets as:

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right), \quad a,b \in R, \quad a \neq 0.$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}, b = nb_0a_0^{-k}, a_0 > 1, b_0 > 0$, n and k are positive integers, we have the following family of discrete wavelets: $\psi_{k,n}(t) = |a|^{-\frac{1}{2}}\psi(a_0^k t - nb_0)$ where $\psi_{k,n}(t)$ form a basis of $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$ then $\psi_{k,n}(t)$ forms an orthonormal basis.

Legendre wavelets $\psi_{nm}(t) = \psi(k,\hat{n},m,t)$ have four arguments: $\hat{n} = 2n-1$, $n = 1,2,3\ldots,2^{k-1}$, k can assume any positive integer, m is the order of Legendre polynomials and t is the normalized time. They are defined on the interval $[0,1)$ as

$$\psi_{nm}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}2^{\frac{k}{2}}P_m(2^k t - \hat{n}), & \text{for } \frac{\hat{n}-1}{2^k} \leqslant t \leqslant \frac{\hat{n}+1}{2^k}, \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

where $m = 0,1,2,\ldots,M-1$, $n = 1,2,3,\ldots,2^{k-1}$. The coefficient $\sqrt{m+\frac{1}{2}}$ is for orthonormality, the dilation parameter is $a = 2^{-k}$ and translation parameter is $b = \hat{n}2^{-k}$.

Here $P_m(t)$ are well-known Legendre polynomials of order m which are defined on the interval $[-1,1]$, and can be determined with the aid of the following recurrence formulae:

$$P_0(t) = 1, \quad P_1(t) = t$$

$$P_{m+1}(t) = \left(\frac{2m+1}{m+1}\right)tP_m(t) - \left(\frac{m}{m+1}\right)P_{m-1}(t), \quad m = 1,2,3,\ldots$$

2.2. Function approximation

A function $f(t)$ defined over $[0,1)$ may be expanded as

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(t) \tag{3}$$

where $c_{nm} = \langle f(t), \psi_{nm}(t)\rangle$, in which $\langle .,. \rangle$ denotes the inner product. If the infinite series in Eq. (3) is truncated, then Eq. (3) can be written as

$$f(t) \cong \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) = C^T\psi(t),$$

where C and $\psi(t)$ are $2^{k-1}M \times 1$ matrices given by

$$C = [c_{10}, c_{11}, \ldots c_{1M-1}, c_{20}, \ldots, c_{2M-1}, \ldots, c_{2^{k-1}0}, \ldots, c_{2^{k-1}M-1}]^T, \tag{4}$$

$$\psi(t) = [\psi_{10}(t), \psi_{11}(t), \ldots, \psi_{1M-1}(t), \psi_{20}(t), \ldots, \psi_{2M-1}(t), \ldots, \psi_{2^{k-1}0}(t), \ldots, \psi_{2^{k-1}M-1}(t)]^T. \tag{5}$$