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ORIGINAL ARTICLE

# Large amplitude free vibration of axially loaded beams resting on variable elastic foundation

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**Abstract** In the present study, large amplitude free vibration of beams resting on variable elastic foundation is investigated. The Euler–Bernoulli hypothesis and the Winkler model have been applied for beam and elastic foundation, respectively. The beam is axially loaded and is restrained by immovable boundary conditions, which yields stretching during vibrations. The energy method and Hamilton's principle are used to derive equation of motion, where after decomposition an ordinary differential equation with cubic nonlinear term is obtained. The second order homotopy perturbation method is applied to solve nonlinear equation of motion. An explicit amplitude-frequency relation is achieved from solution with relative error less than 0.07% for all amplitudes. This solution is applied to study effects of variable elastic foundation, amplitude of vibration and axial load on nonlinear frequency of beams with simply supported and fully clamped boundary conditions. Proposed formulation is capable to dealing with any arbitrary distribution of elastic foundation.


## 1. Introduction

Beams are fundamental component in engineering and have wide applications in design and fabrication of structures and machines such as tall building, huge cranes, bridges, turbine and compressor blades. They are also used as simple and accurate model for analysis of complex engineering structures. Natural frequencies and dynamic response of beam-like structure in small amplitude vibration studied by many researchers through analytical and numerical methods and different aspects have been considered. Mathematical model of small amplitude vibration is in the form of linear differential equation which is relatively simple for handling. Nowadays, the demand for light-weight structures and machines is continuously increasing. These light-weight systems are more flexible due to their high aspect ratio and external excitation such as wind load causing large amplitude vibration in them. Large amplitude vibration induced nonlinear terms in differential equation of motion. In the case of beam with immovable ends, axial stretching of the beam during vibration with large amplitude is the source of nonlinearity. The nonlinear vibration of beams due to the large amplitude of vibration has received considerable attention by many researchers. Bhashyam and Prathap [1] presented a Galerkin finite element method for

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studying nonlinear vibration of beams describable in terms of moderately large bending theory. Ozkaya et al. [2] study the nonlinear response of a beam-mass system with immovable ends by applying the method of multiple scales directly to partial differential equation of motion. Gayesh and Balar [3] study nonlinear parametric vibration and stability of axially moving viscoelastic Rayleigh beams, and they derived the partial-differential equation of motion for large amplitude vibration through geometrical, constitutive, and dynamical relations. Abdel-Jaber et al. [4] study nonlinear frequencies of an elastically restrained tapered beam. They used the nonlinear curvature and the axial shortening due to transverse deflection in the energy formulation of system. Merrimi et al. [5] investigate the geometrically nonlinear steady state periodic forced response of a clamped–clamped beam containing an open crack. The crack has been modeled as a linear spring in which, for a given depth, the spring constant remains the same for both directions. Sedighi et al. [6] derive an analytic solution of transversal oscillation of quintic nonlinear beam with homotopy analysis method. Baghani et al. [7] study large amplitude free vibrations and post-buckling of unsymmetrically laminated composite beams on nonlinear elastic foundation. Lai et al. [8] derive the analytical solutions for large amplitude vibration of thin functionally graded beams. Sedighi and Shirazi [9] study effect of deadzone nonlinear boundary condition on large amplitude vibration of cantilever beam. There are many other researches exist on buckling, linear and nonlinear vibrations of beams and plates [10–18].

It is obvious that the accurate analysis of structures required an understanding of soil–structure interaction. The surrounding soil increases resistance of buried structures and significantly changes modal parameters of them. Many practical cases in engineering related to soil–structure interaction can be modeled by means of a beam on elastic foundation. The well-known model for elastic foundations is Winkler. The Winkler model of elastic foundation is the most preliminary in which the vertical displacement is assumed to be proportional to the contact pressure at an arbitrary point [19], in another words, the foundation modeled as a series of closely spaced and mutually independent linear elastic springs. Different problems of beams resting on elastic foundation were studied and reported in the literature [20–25]. Often, researcher assumed that the foundation has constant value through the length of the beam and only limited studies exist for dynamic analysis of beams on variables foundations. Eisenberger and Clastornik [26] study free vibration and buckling of the Euler–Bernoulli beams on variable Winkler foundation. Zhou [27] presents a general solution to vibration of the Euler–Bernoulli beams on variable elastic foundation. He assumed the reaction force of the foundation on the beam as the external force acting on the beam. Pradhan and Murmu [28] study thermo-mechanical vibration of sandwich beams resting on variable Winkler foundation using differential quadrature method. Kacar et al. [29] apply differential transform method to investigate free vibration of the Euler–Bernoulli beams on variable Winkler foundation.

According to literature survey, large amplitude free vibration of beam resting on variable elastic foundation has not been studied and for the first time has been studied in this paper. Equation of motion is obtained from energy method by invoking Hamilton principle, and then homotopy perturbation method [30–33] is applied to solve governing nonlinear differential equation. Comparisons are made with studies in the open literature in which special cases of the present problem have been studied and very good agreement observed. Finally, some new and more useful results are extracted from the present formulation.

## 2. Mathematical formulation

Consider a straight beam under axial load which resting on variable Winkler foundation as shown in Fig. 1. The beam has length $L$, rectangular cross section with the area of $S$, the cross-sectional moment of inertia of $I$ and thickness of $h$. The beam was made from isotropic material with $E$ as modulus of elasticity and $\rho$ as mass per unit volume. Stiffness of Winkler foundation changes along the beam length and is the function of spatial coordinate along the beam length (i.e. $\bar{x}$). Using Euler–Bernoulli beam hypothesis, the strain energy induced by large displacement amplitude is given by the following:

$$\Pi = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 \bar{y}(\bar{x},\bar{t})}{\partial \bar{x}^2}\right)^2 d\bar{x} + \frac{1}{2}\int_0^L ES\left(\frac{\partial \bar{u}(\bar{x},\bar{t})}{\partial \bar{x}} + \frac{1}{2}\left(\frac{\partial \bar{y}(\bar{x},\bar{t})}{\partial \bar{x}}\right)^2\right)^2 d\bar{x} + \frac{1}{2}\int_0^L k(\bar{x})\bar{y}(\bar{x},\bar{t})^2 d\bar{x} \tag{1}$$

where $\bar{u}$ and $\bar{y}$ are axial and transverse displacements, respectively. $k(\bar{x})$ is the mathematical expression for variable Winkler foundation. The kinetic energy is given by the following:

$$T = \frac{1}{2}\int_0^L \rho S\left(\frac{\partial \bar{y}(\bar{x},\bar{t})}{\partial \bar{t}}\right)^2 d\bar{x} \tag{2}$$

The external work done by axial load can be written as follows:

$$W = \frac{f}{2}\int_0^L \left(\frac{\partial \bar{y}(\bar{x},\bar{t})}{\partial \bar{x}}\right)^2 d\bar{x} \tag{3}$$

Using the Lagrangian of the system and invoking Hamilton's principle, we have the following:

$$\delta\int_{t_1}^{t_2} (T - \Pi + W)d\bar{t} = 0 \tag{4}$$

Substituting Eqs. (1)–(3) into Eq. (4), performing the necessary algebra and eliminating axial displacement, the following partial differential equation is obtained for motion:

$$EI\frac{\partial^4 \bar{y}(\bar{x},\bar{t})}{\partial \bar{x}^4} + \rho A\frac{\partial^2 \bar{y}(\bar{x},\bar{t})}{\partial \bar{t}^2} + f\frac{\partial^2 \bar{y}(\bar{x},\bar{t})}{\partial \bar{x}^2} + k(\bar{x})\bar{y}(\bar{x},\bar{t}) - \frac{EA}{2L}\frac{\partial^2 \bar{y}(\bar{x},\bar{t})}{\partial \bar{x}^2}\int_0^L \left(\frac{\partial \bar{y}(\bar{x},\bar{t})}{\partial \bar{x}}\right)^2 d\bar{x} = 0 \tag{5}$$

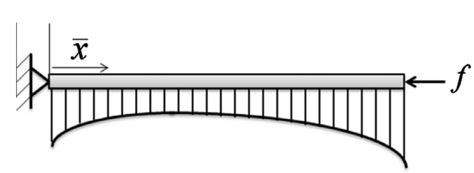


**Figure 1** Schematic of the beam under axial load and resting on variable elastic foundation.