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Application of perturbation–iteration method to Lotka–Volterra equations

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Abstract Perturbation–iteration method is generalized for systems of first order differential equations. Approximate solutions of Lotka–Volterra systems are obtained using the method. Comparisons of our results with each other and with numerical solutions are given. The method is implemented in *Mathematica*, a major computer algebra system. The package *PerturbationIteration.m* automatically carries out the tedious calculations of the method.

1. Introduction

The study of methods for approximate solutions of nonlinear models in real life has always been a growing branch of applied mathematical sciences. Many methods with different capabilities and limitations have been developed. The well-known approximate analytical method, the perturbation technique [1] can deal with weakly nonlinear systems due to the small parameter assumption. To overcome this limitation, methods such as the linearized perturbation method [2], the Lindstedt–Poincaré method with modified frequency expansion [3], the multiple-scale Lindstedt–Poincaré method [4] and the parameter expanding method [5] were developed. Also methods such as the Adomian decomposition method [6], the variational iteration method [7], and the homotopy analysis method [8] were among the non-perturbative methods that were applied to many interesting mathematical problems.

Other attempts to treat both weakly and strongly nonlinear problems were through iteration procedures which used preformed alternative equations to obtain approximate solutions. Just to list a few, He [9] linearized the nonlinear terms by substitution of iterative solution functions from previous iteration results, Mickens' iteration procedure [10] was for specific problems, and variational iteration method [11] was used to solve boundary value problems.

Recently, the perturbation–iteration method, which gives valuable solutions for strongly nonlinear problems [12,13] has been developed. In [14], approximate solutions of some nonlinear heat transfer problems were obtained, and the comparison of the results showed that perturbation–iteration method fits better than the variational iteration method as the parameter measuring the nonlinearity takes larger values.

The aim of this study was to develop perturbation–iteration algorithms for systems of first order differential equations and to obtain accurate solutions of Lotka–Volterra differential equations. Many analytical methods such as Adomian decomposition [15], variational iteration method [16], homotopy analysis method [17], and optimal parametric iteration method [18] were successfully applied to this type of problems.

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The paper is organized as follows. In Section 2, the perturbation–iteration algorithm for systems is explained. The method is applied to Lotka–Volterra equations in Section 3. In Section 4, comparisons of our results with other methods are given. The package *PerturbationIteration.m* is described in Section 5. Concluding remarks are given in Section 6.

2. Perturbation–iteration method for systems of first order differential equations

In this section, the perturbation–iteration method, described in [12,13], is generalized toward systems of first order differential equations. The generalization is developed by taking arbitrary number of terms in Taylor series expansion and one correction term in perturbation expansion PIA(1,M) i.e. the first number expresses the correction terms in the perturbation expansion and the second number expresses the derivative orders in the Taylor expansion. Consider the following system of first order differential equations:

$$F_k(\dot{u}_k, u_j, \varepsilon, t) = 0, \quad j = 1, 2, \ldots, K, \; k = 1, 2, \ldots, K, \tag{1}$$

where u_j are the dependent variables, t is the independent variable, K is the number of dependent variables, ε is the artificially introduced small parameter and the dot stands for the derivative. Reconsider Eq. (1) in the following iterative form:

$$F_k(\dot{u}_{k,n+1}, u_{j,n+1}, \varepsilon, t) = 0, \tag{2}$$

where the subscript n expresses the number of iterations completed in the procedure. More clearly, the system of equations is given below

$$\begin{aligned} &F_1 = F_1(\dot{u}_1, u_1, u_2, \ldots, u_K, \varepsilon, t) \\ &F_2 = F_2(\dot{u}_2, u_1, u_2, \ldots, u_K, \varepsilon, t) \\ &\vdots \\ &F_K = F_K(\dot{u}_K, u_1, u_2, \ldots, u_K, \varepsilon, t). \end{aligned} \tag{3}$$

Next, define the following iterative perturbation series:

$$u_{j,n+1} = u_{j,n} + \varepsilon u^c_{j,n}, \tag{4}$$

Taylor series expansion of equation (1) is given as follows:

$$F_k = \sum_{m=0}^{M} \frac{1}{m!}\left[\left(\frac{d}{d\varepsilon}\right)^m F_k\right]_{\varepsilon=0} \varepsilon^m, \quad k = 1, 2 \ldots\ldots K \tag{5}$$

with the derivative operator defined as follows:

$$\frac{d}{d\varepsilon} = \frac{\partial \dot{u}_{k,n+1}}{\partial \varepsilon}\frac{\partial}{\partial \dot{u}_{k,n+1}} + \sum_{j=1}^{K}\left(\frac{\partial u_{j,n+1}}{\partial \varepsilon}\frac{\partial}{\partial u_{j,n+1}}\right) + \frac{\partial}{\partial \varepsilon}. \tag{6}$$

Calculating derivatives at $\varepsilon = 0$ and substituting Eq. (6) into Eq. (5) yields

$$F_k = \sum_{m=0}^{M} \frac{1}{m!}\left\{\left(\dot{u}^c_{k,n}\frac{\partial}{\partial \dot{u}_{k,n+1}} + \sum_{j=1}^{K} u^c_{j,n}\frac{\partial}{\partial u_{j,n+1}} + \frac{\partial}{\partial \varepsilon}\right)^m F_k\right\}_{\varepsilon=0} \varepsilon^m = 0,$$

$$k = 1, 2 \ldots\ldots K \tag{7}$$

Note that, by taking larger values of M, i.e., increasing the number of terms in the Taylor series expansion, one can develop different algorithms for the specific problem with the help of Eq. (7). Since one correction term in the perturbation expansion and m'th order derivatives in the Taylor expansion are taken, the algorithm is PIA(1,m). A more general algorithm with PIA(n,m) can be constructed but would cause too much complexity in the applications.

3. Application to Lotka–Volterra problems

The Lotka–Volterra equations, also known as the *predator–prey equations*, are frequently used by mathematicians to describe the time evolution of the species in the dynamics of biological systems [15–17]. In the more general framework one can study the following multidimensional Lotka–Volterra system:

$$\frac{du_k}{dt} = u_k\left(b_k + \sum_{j=1}^{K} a_{kj}u_j\right), \; u_k(0) = c_k, \quad k = 1, 2, \ldots, K, \tag{8}$$

where c_k represent the populations of the species at the beginning of the evolution. In the following subsections, solutions of one dimensional and multidimensional systems such as $K = 1$, 2, 3,... are investigated by the perturbation–iteration method.

3.1. One dimensional system ($K = 1$)

One dimensional Lotka–Volterra equation, known as the Verhulst equation [19], describes the behavior of population in time of one species competing for a given finite source of food:

$$\frac{du}{dt} = u(b + au), \quad b > 0, \quad a < 0, \quad u(0) > 0. \tag{9}$$

The exact solution of (9) is

$$u(t) = \begin{cases} \dfrac{be^{bt}}{\frac{b+au(0)}{u(0)} - ae^{bt}}, & b \neq 0 \\ \dfrac{u(0)}{1-au(0)t}, & b = 0. \end{cases} \tag{10}$$

Approximate solutions of Eq. (9) will be obtained using two different perturbation–iteration algorithms by taking $M = 1$ and $M = 2$.

3.1.1. Perturbation iteration method with first order derivatives in the Taylor series expansion PIA(1,1)

First rewrite Eqs. (9) and (3) in the following form:

$$F_1 = \dot{u}_{1,n+1} - bu_{1,n+1} - \varepsilon a u^2_{1,n+1} = 0, \tag{11}$$

$$u_{1,n+1} = u_{1,n} + \varepsilon u^c_{1,n}, \tag{12}$$

where ε is artificially introduced as a small parameter. Reorganizing Eq. (7) for Eq. (9) yields

$$\frac{du^c_{1,n}}{dt} + \frac{\partial F_1/\partial u_{n+1}}{\partial F_1/\partial \dot{u}_{n+1}} u^c_{1,n} = -\frac{\varepsilon F_1}{\partial F_1/\partial \dot{u}_{n+1}} - \frac{\partial F_1/\partial \varepsilon}{\partial F_1/\partial \dot{u}_{n+1}}, \tag{13}$$

where the derivatives are

$$F_1|_{\varepsilon=0} = \dot{u}_n - bu_n, \quad \left.\frac{\partial F_1}{\partial u_{n+1}}\right|_{\varepsilon=0} = -b, \quad \left.\frac{\partial F_1}{\partial \dot{u}_{n+1}}\right|_{\varepsilon=0} = 1,$$

$$\left.\frac{\partial F_1}{\partial \varepsilon}\right|_{\varepsilon=0} = -au^2_n. \tag{14}$$

Finally, the iteration equation is obtained by substituting (14) into (13) and setting $\varepsilon = 1$:

$$\frac{du^c_{1,n}}{dt} - bu^c_{1,n} = -\dot{u}_{1,n} + bu_{1,n} + au^2_{1,n}. \tag{15}$$