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SHORT COMMUNICATION

Exact solution of unsteady flow generated by sinusoidal pressure gradient in a capillary tube

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Abstract In this paper, the mathematical modeling of unsteady second grade fluid in a capillary tube with sinusoidal pressure gradient is developed with non-homogenous boundary conditions. Exact analytical solutions for the velocity profiles have been obtained in explicit forms. These solutions are written as the sum of the steady and transient solutions for small and large times. For growing times, the starting solution reduces to the well-known periodic solution that coincides with the corresponding solution of a Newtonian fluid. Graphs representing the solutions are discussed.

1. Introduction

Generally, many engineering fluids, e.g. dilute polymer, pastes, slurries, synovial, paints exhibit numerous strange features, e.g. shear loss/thickening and display of elastic effects which cannot be well described by the Navier–Stokes equations. Various rheological models have been proposed to portray their non-Newtonian flow behavior. The fluids of a differential type have acquired special status due to their elegance, Dunn and Rajagopal [1]. One such type of rheological model is the differential type fluid model and second grade fluid is one of the subclasses of these differential type fluid models. Due to its ability in successfully capturing various non-Newtonian effects, it has been the subject of many investigations [2–7], etc.

Recently, some of the newly developed approximate analytical tools have been employed by various researchers to solve several basic flow problems of second grade fluid in cylindrical geometry, and the approximate solutions were found for the velocity profiles [8–11]. All these quoted analyses of the fluid flow take place due to the drag of boundary in a bath of fluid. However, no solution expressions were obtained for the flow rate that is solely due to the oscillating pressure gradient. The task of the present paper is to venture further in this regime. For what we are interested to examine the unsteady second grade fluid in a capillary round tube driving by a sinusoidal pressure gradient. Due to the complexity of the governing equations, finding accurate solutions is not easy. Therefore, we made an attempt to obtain an exact solution to the differential equation. A solution for the velocity field is derived as the sum of steady and transient solutions, describing the

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motion of the fluid for small and large times using exact analysis. This review would serve as an important reference for researchers in this area.

2. Formulation of the problem

Consider an incompressible, laminar, viscoelastic fluid pulsating flow in a capillary tube with a radius of r_0 driven by a pressure gradient that varies sinusoidally with time as

$$\nabla p = \mathbf{e}_z(B_0 + B_1 \exp(i\omega t)). \tag{1}$$

where the pressure gradient contains a steady and a pulsating part, of amplitudes B_0 and B_1, respectively. The unit vector $\mathbf{e}_z$ is in the z-direction parallel to the flow, ω is the frequency of the pressure gradient, t is the time and $i = \sqrt{-1}$ is the imaginary constant. Using the pressure gradient given in Eq. (1), the cosine and sine oscillations can be treated by taking the real and imaginary parts of the pressure gradient ∇p. Fig. 1 shows the physical configuration.

The Cauchy stress tensor $\mathbf{T}$ for an incompressible homogeneous second grade fluid is given by the constitutive equations

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} = \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{2}$$

where $\mathbf{I}$ is the identity tensor, p is the pressure, $\mathbf{S}$ is the extra-stress tensor, μ is the dynamic viscosity, and α_1, α_2 are normal stress moduli and $\mathbf{A}_1$ and $\mathbf{A}_2$ are the kinematic tensors defined as

$$\mathbf{A}_1 = (\text{grad}\ \mathbf{u}) + (\text{grad}\ \mathbf{u})^T, \tag{3}$$

$$\mathbf{A}_2 = \frac{d}{dt}\mathbf{A}_1 + \mathbf{A}_1(\text{grad}\ \mathbf{u}) + (\text{grad}\ \mathbf{u})^T\mathbf{A}_1, \tag{4}$$

where $\frac{d}{dt}$ is the material time derivative and $\mathbf{u}$ is velocity vector.

The fluid velocity through capillary tube is moving with velocity of the form

$$\mathbf{u} = \mathbf{u}(r,t) = u(r,t)\mathbf{e}_z, \tag{5}$$

where $\mathbf{e}_z$ is the unit vector along z-axis.

Introducing Eq. (5) into Eq. (2), we find that

$$\mathbf{T}_{r,z} = \left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\frac{\partial u(r,t)}{\partial r}, \tag{6}$$

By considering the pressure gradient in the axial direction, the balance of the linear momentum in the absence of body forces leads to the following equation

$$\rho\frac{\partial u}{\partial t} = (B_0 + B_1\exp(i\omega t)) + \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\mathbf{T}_{r,z} \tag{7}$$

Eliminating $\mathbf{T}_{r,z}$ between Eqs. (6) and (7), we obtain

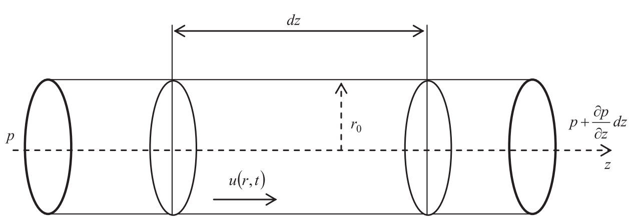

Figure 1 The physical configuration.

$$\rho\frac{\partial u}{\partial t} = (B_0 + B_1\exp(i\omega t)) + \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) + \alpha_1 \times \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \tag{8}$$

The initial and boundary conditions are

$$u = 0 \text{ at } t = 0, \quad \text{for } 0 \leqslant r \leqslant r_0, \tag{9}$$

$$\frac{\partial u}{\partial r} = 0 \text{ at } r = 0, \quad \text{for all } t \geqslant 0, \tag{10}$$

$$u = 0 \text{ at } r = r_0, \quad \text{for all } t \geqslant 0. \tag{11}$$

Consider the following dimensionless quantities

$$u^* = \frac{u}{u_m}, \quad \alpha^* = \frac{\alpha_1}{\rho r_0^2}, \quad t^* = \frac{\mu t}{\rho r_0^2}, \quad r^* = \frac{r}{r_0}, \quad \omega^* = \frac{\omega r_0^2}{\upsilon}. \tag{12}$$

we obtain the dimensionless initial-boundary values problem (dropping $*$ the notation)

$$\frac{\partial u}{\partial t} = \gamma(B_0 + B_1\exp(i\omega t)) + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \alpha\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \tag{13}$$

$$u = 0 \text{ at } t = 0, \quad \text{for } 0 \leqslant r \leqslant 1, \tag{14}$$

$$\frac{\partial u}{\partial r} = 0 \text{ at } r = 0, \quad \text{for all } t \geqslant 0, \tag{15}$$

$$u = 0 \text{ at } r = 1, \quad \text{for all } t \geqslant 0. \tag{16}$$

where $\gamma = \frac{r_0^2}{\mu u_m}$ is a constant that controls the amplitude of the pressure fluctuation and $u_m = \frac{r_0^2}{\mu}\left(\widehat{\frac{\partial p}{\partial z}}\right)$ is the cross-sectional mean velocity for the time-averaged flow.

3. Solution technique

3.1. Steady solution

Assume that the solution to Eq. (13) is of the form

$$u(r,t) = u_s(r) + u_t(r,t), \tag{17}$$

where u_s is a steady solution and u_t is the transient solution component. Note that, if we allow $t \to \infty$, we obtain the steady solution.

Substituting Eq. (17) into (13) we have

$$\frac{\partial u_s}{\partial t} + \frac{\partial u_t}{\partial t} = \gamma(B_0 + B_1\exp(i\omega t)) + \left(\frac{\partial^2 u_s}{\partial r^2} + \frac{\partial^2 u_t}{\partial r^2} + \frac{1}{r}\frac{\partial u_s}{\partial r} + \frac{1}{r}\frac{\partial u_t}{\partial r}\right) + \alpha\frac{\partial}{\partial t}\left(\frac{\partial^2 u_t}{\partial r^2} + \frac{1}{r}\frac{\partial u_t}{\partial r}\right). \tag{18}$$

Considering $\frac{\partial u_s}{\partial t} = 0$, Eq. (18) can be separated into two equations

$$\frac{\partial^2 u_s}{\partial r^2} + \frac{1}{r}\frac{\partial u_s}{\partial r} = -\gamma B_0, \tag{19}$$

$$\frac{\partial^2 u_t}{\partial r^2} + \frac{1}{r}\frac{\partial u_t}{\partial r} + \alpha\frac{\partial}{\partial t}\left(\frac{\partial^2 u_t}{\partial r^2} + \frac{1}{r}\frac{\partial u_t}{\partial r}\right) - \frac{\partial u_t}{\partial t} = -\gamma B_1\exp(i\omega t), \tag{20}$$