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ORIGINAL ARTICLE

Unsteady rotational flows of an Oldroyd-B fluid due to tension on the boundary

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Abstract Unsteady Taylor–Couette flows of an Oldroyd-B fluid, which fills a straight circular cylinder of radius R, are studied. Flows are generated by the oscillating azimuthal tension which is given on the cylinder surface. As a novelty, authors used in this paper the governing equation related to the tension field. The closed forms of the shear stress and velocity fields corresponding to the flow problems are obtained by means of the integral transforms method. Expressions for the azimuthal tension and fluid velocity were written as sums between the "permanent component" (the steady-state component) and the transient component. By customizing values of parameters from the mathematical model were obtained the corresponding solutions of other types of fluids, namely, Maxwell fluids. By using numerical simulations and diagrams of the azimuthal stress, the fluid behavior has been analyzed. The necessary time to achieve the "steady-state" was, also, determined.

1. Introduction

The Oldroyd-B fluid model is very important among the fluids of rate type due to its special behavior. Also, this model contains the Newtonian fluid model and Maxwell fluid model as special cases. The Oldroyd-B fluid model [1,2] considers the memory effects and elastic effects exhibited by a large class of fluids, such as the biological and polymeric liquids. Guillope and Saut [3] and Fontelos and Friedman [4] established the stability, existence and uniqueness results for some shearing flows of such fluids. Exact solutions for some simple flows of Oldroyd-B fluids were presented by many authors, See, for example, Rajagopal and Bhatnagar [5], Hayat et al. [6,7]. Recently, various problems regarding flows of Oldroyd-B fluids through cylindrical domains have been studied. Singh and Varshney [8] have considered the unsteady laminar flow of an electrically conducting Oldroyd fluid through a circular cylinder boundary by permeable bed under the influence of an exponentially decreasing pressure gradient in porous medium. Burdujan [9] studied Taylor–Couette flows of the Oldroyd-B fluid with fractional derivatives within the annular region between two infinitely coaxial circular cylinders due to a time-dependent axial tension given on the surface of the inner cylinder. The unsteady unidirectional transient flow of Oldroyd-B fluid with fractional time derivatives, in an annular domain, produced by a constant pressure gradient and a translation with constant velocity of the inner cylinder was studied by Mathur and Khandelwal [10]. Liu et al. [11] studied some helical flows of an Oldroyd-B fluid with time-fractional

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derivatives, between two infinite concentric oscillating cylinders and within an infinite circular oscillating cylinder. Most existing solutions in the literature correspond to problems with boundary conditions on the velocity. There are several practical problems with the specified force on the boundary [12–14]. For example in [12], Renardy has studied the motion of a Maxwell fluid across a strip bounded by parallel plates and proved that, in order to formulate a well posed problem it is necessary to impose the boundary conditions on the stresses at the inflow boundary. In [13], Renardy explained how well posed boundary value problems can be formulated using boundary conditions on stresses. Waters and King [15] were among the first specialists who used the shear stress on the boundary to find exact solutions for motions of rate type fluids. Other interesting problems regarding flows of non-Newtonian fluids, in various geometry or boundary conditions, can be finding in the references [17–23]. Our goal is to investigate unsteady flows of Oldroyd-B fluids in an infinite circular cylinder. In the present paper the governing equation of the flow is related to the azimuthal tension and we considered the boundary conditions on the shear stress. The flow of the fluid is due to rotation of the cylinder around its axis, under the action of oscillating shear stress $fH(t)\sin(\omega t)$ or $fH(t)\cos(\omega t)$ given on the boundary. Finally, solutions of the Maxwell fluid flows are obtained as particular cases of our general results. Also the comparison between models is underlined by graphical illustrations.

2. Problem formulation

The constitutive equations for an Oldroyd-B fluid [1] are

$$\begin{aligned}\mathbf{T} = & \ -p\mathbf{I} + \mathbf{S}, \mathbf{S} + \lambda\left(\frac{d\mathbf{S}}{dt} - \mathbf{LS} - \mathbf{SL}^T\right) \\ = & \ \mu\left\{\mathbf{A} + \lambda_r\left(\frac{d\mathbf{A}}{dt} - \mathbf{LA} - \mathbf{AL}^T\right)\right\},\end{aligned} \tag{1}$$

where $\mathbf{T}$ is Cauchy stress tensor, $-p\mathbf{I}$ is indeterminate spherical stress, $\mathbf{S}$ is extra stress tensor, $\mathbf{L}$ is velocity gradient, μ is the dynamic viscosity, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ is first the Rivilin–Erickson tensor, λ and λ_r $(0 \leqslant \lambda_r < \lambda)$ are relaxation and retardation time. Assume an infinite circular cylinder of radius R with axis of rotation along z-axis. Cylinder is filled with an Oldroyd-B fluid which is at rest at time $t = 0$. After time $t = 0^+$ the cylinder applies an oscillating rotational shear stress $fH(t)\sin(\omega t)$ or $fH(t)\cos(\omega t)$ to the fluid, $f > 0$ is constant and ω is the angular frequency of oscillations. We assume that, the fluid is incompressible and homogeneous. Furthermore we assume that velocity field and extra-stress tensor are of the form

$$\mathbf{V} = \mathbf{V}(r,t) = w(r,t)\mathbf{e}_\theta, \quad \mathbf{S} = \mathbf{S}(r,t), \tag{2}$$

where $\mathbf{e}_\theta$ is the unit vector in the θ direction of the cylindrical coordinate system. Since the fluid and the cylinder are at rest at time $t = 0$, therefore,

$$w(r,0) = 0, \ \mathbf{S}(r,0) = \mathbf{0}. \tag{3}$$

Introducing (2) in $(1)_2$ and by using (3) we get $S_{rr} = S_{rz} = S_{z\theta} = S_{zz} = 0$, along with the following meaningful partial differential equation

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\tau(r,t) = \mu\left(1 + \lambda_r\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)w(r,t), \tag{4}$$

where $\tau(r,t) = S_{r\theta}(r,t)$ is one of the nonzero component of extra stress tensor. The balance of linear momentum in the absence of body forces reduces to [8]

$$\rho\frac{\partial w(r,t)}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\tau(r,t), \tag{5}$$

$\tau_{st}(r,t)$ ρ being the constant density of the fluid. By eliminating $w(r,t)$ between Eqs. (4) and (5) we get the following governing equation for the shear stress [24]

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\frac{\partial\tau(r,t)}{\partial t} = \nu\left(1 + \lambda_r\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2}\right)\tau(r,t), \tag{6}$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity of the fluid. The appropriate initial and boundary conditions are

$$\tau(r,t)|_{t=0} = \left.\frac{\partial\tau(r,t)}{\partial t}\right|_{t=0} = 0, \tag{7}$$

$$\tau(R,t) = fH(t)\sin\omega t \ \ or \ \ \tau(R,t) = fH(t)\cos\omega t, \tag{8}$$

$H(t)$ being the Heaviside unit step function. Converting our problem (6)–(8) into the complex field ($\tau = \tau_c + i\tau_s$ with τ_c and τ_s being solutions for cosine, respectively sine boundary conditions), we have

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\frac{\partial\tau(r,t)}{\partial t} = \nu\left(1 + \lambda_r\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2}\right)\tau(r,t) \tag{9}$$

$$\tau(r,t)|_{t=0} = \left.\frac{\partial\tau(r,t)}{\partial t}\right|_{t=0} = 0 \tag{10}$$

$$\tau(R,t) = fH(t)e^{i\omega t}, \quad f > 0. \tag{11}$$

By introducing the following dimensionless quantities

$$t^* = \frac{t}{\lambda}, \ r^* = \frac{r}{R}, \ w^* = \frac{w}{U_0}, \ \tau^* = \frac{\tau}{f}, \ U_0 = \frac{\lambda f}{\rho R}, \beta^* = \frac{\lambda_r}{\lambda}, \ \omega^* = \lambda\omega, \tag{12}$$

Eqs. (5), (9)–(11) becomes (dropping the star notation)

$$\frac{\partial w(r,t)}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\tau(r,t), \tag{13}$$

$$Re\left(1 + \frac{\partial}{\partial t}\right)\frac{\partial\tau(r,t)}{\partial t} = \left(1 + \beta\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2}\right)\tau(r,t), \tag{14}$$

$$\tau(r,t)|_{t=0} = \left.\frac{\partial\tau(r,t)}{\partial t}\right|_{t=0} = 0, \ \ w(r,0) = 0 \tag{15}$$

$$\tau(1,t) = H(t)e^{i\omega t}, \tag{16}$$

where $Re = \frac{R^2}{\lambda\nu}$ is the Reynolds number.

3. Solution of the problem

In order to determine the exact analytical solution, we shall use the Laplace and finite Hankel transforms [25]. By applying the temporal Laplace transform to Eqs. (14) and (16) and using the initial conditions (15) we get the following transformed forms,

$$Re(1+q)q\bar{\tau}(r,q) = (1+\beta q)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2}\right)\bar{\tau}(r,q), \tag{17}$$