



Sextic and decatic anharmonic oscillator potentials: Polynomial solutions

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ARTICLE INFO

Keywords:

Anharmonic oscillator
Schrödinger equation
Energy eigenvalues
Polynomial potentials
Polynomial solutions
Exact solutions

ABSTRACT

We seek the possible polynomial solutions of the Schrödinger equation for the sextic and decatic potentials. Under certain conditions on the parameters of the potentials, we show that these potentials are exactly solvable. We evaluate the first four eigenstates for both potentials. We derive general expressions of the energy levels, for high energy levels, eigenvalues are a function of potentials' parameters and the eigenfunction's zeros.

1. Introduction

Anharmonic oscillators have an important role in the evolution of many branches of physics. In fact, their importance is due to the anharmonic nature of many quantum systems vibrations. The number of quantum systems, whose Schrödinger equation has exact solutions is very limited such as harmonic oscillator and hydrogen atom. For complicated potential energy problems, many numerical methods were developed [1–10]. To determine the ground state eigenvalues and eigenfunctions for polynomial potentials, A. De Freitas et al. [11] used an extended two-point quasi-rational approximation technique. However, G. M. Gayathri et al. [12] applied the Ginsberg-Montroll method. In order to calculate both the wave functions and the energy eigenvalues for the ground and first excited states of the quartic, sextic and octic potentials with high precision. P. Amore et al. [13] performed a comparative study of quantum anharmonic potentials.

Furthermore, F.M. Fernández and H. Gıftci [14] accomplished alternative perturbation expansions for the sextic anharmonic oscillator. Recently, the analytic (polynomial) solutions have been introduced to investigate Schrödinger equation solutions for confined and unconfined quantum systems [15–20]. In fact, D. Brandon and N. Saad [21] showed that under certain conditions on the potential's parameters, the decatic polynomial potential was exactly solvable and for arbitrary values of the potential parameters, the asymptotic iteration method was presented. Moreover, if these solutions are available, they constitute very important tools to check and improve numerical solutions presented to solve complicated physical systems. The solvability of such problems is directly

tied to the finite-degree polynomial representation of the potential and wavefunctions that decay exponentially.

In this paper, before introducing the polynomial structure, we formulate the problem. Then, to make paper easy to read, we briefly recall the asymptotic iteration method in the second section. Here, we perform the necessary and sufficient conditions for the corresponding energy-dependent polynomial solutions. On the following section, we solve the well-known harmonic oscillator potential problem. In Section 4, we study the sextic anharmonic potential problem, the first five energies levels values are calculated. Section 5 concerns decatic anharmonic potential case. Finally, the conclusion is given in Section 6.

2. Problem formulation

2.1. Schrödinger equation

Solving the one-dimensional time-independent Schrödinger equation is a fundamental and primary step in order to calculate the energy eigenvalues for anharmonic oscillator potentials. The time-independent Schrödinger equation may be written as:

$$-\frac{\hbar^2}{2m} \frac{d^2\psi_n(x)}{dx^2} + V(x)\psi_n(x) = E\psi_n(x) \quad (1)$$

where $V(x) = \alpha_2x^2 + \alpha_4x^4 + \dots + \alpha_{2N}x^{2N}$,

$\alpha_2 = \frac{1}{2}k$,

and $\omega^2 = \frac{k}{m}$,

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$V(x)$ is an anharmonic symmetric potential energy ($\alpha_{2N} > 0$), E the system's energy and ψ_n is the wave function (n th eigenstate). Dividing Eq. (1) by $\hbar\omega$ we find that:

$$-\frac{\hbar}{2\omega m} \frac{d^2\psi_n(x)}{dx^2} + \left(\frac{\alpha_2}{\hbar\omega} x^2 + \frac{\alpha_4}{\hbar\omega} x^4 + \dots + \frac{\alpha_{2N}}{\hbar\omega} x^{2N} \right) \psi_n(x) = \frac{E}{\hbar\omega} \psi_n(x). \quad (2)$$

Putting $\lambda = \frac{\hbar}{2\omega m}$ and $\varepsilon = \frac{E}{\hbar\omega}$ and changing to the variable $y = \left(\frac{2\omega m}{\hbar} \right)^{1/2} x$,

We deduce the dimensionless equation:

$$\frac{d^2\psi_n(y)}{dy^2} + \left(\varepsilon - \left(\frac{\alpha_2}{\hbar\omega} \lambda y^2 + \frac{\alpha_4}{\hbar\omega} \lambda^2 y^4 + \dots + \frac{\alpha_{2N}}{\hbar\omega} \lambda^N y^{2N} \right) \right) \psi_n(y) = 0, \quad (3)$$

and

$$\frac{d^2\psi_n(y)}{dy^2} + (\varepsilon - v(y))\psi_n(y) = 0, \quad (4)$$

where $v(y) = b_2 y^2 + b_4 y^4 + \dots + b_{2N} y^{2N}$,

$$b_{2i} = \frac{\alpha_{2i} \lambda^i}{\hbar\omega},$$

and $i = 2, 4, 6, \dots$

We will examine the polynomial solutions of this linear differential equation. The necessary and sufficient conditions for corresponding energy-dependent polynomial solutions will be given in detail.

2.2. Polynomial solution structure

To solve our problem, firstly, we suppose the solution as:

$$\psi_n(y) = A f_n(y) \exp(-h(y)), \quad (5)$$

where the functions f and h are given by $f_n(y) =$

$$\begin{cases} 1 & \text{for } n = 0 \\ \prod_{j=1}^n (y - y_{jn}) & \text{for } n > 0, \end{cases} \quad y_{jn} \text{ are the eigenstates nodes (wave function's}$$

zeros) and $h(y) = \sum_{p=1}^N a_{2p} y^{2p}$, the coefficients a_{2p} are real and a_{2N} positive, and A is the wave's normalization constant. Secondly, for even potential energy, wave functions are either symmetric or antisymmetric; consequently, their zeros are opposite two by two. The substitution of the wave function ψ_n given by Eq. (5) in Eq. (4), leads to the necessary and sufficient conditions for corresponding energy-dependent polynomial solutions:

$$G_n(y) = \frac{1}{A \exp(-h(y))} \frac{d^2\psi_n(y)}{dy^2} + (\varepsilon - v(y)) f_n(y) = 0. \quad (6)$$

2.3. Asymptotic iteration method

Asymptotic iteration method states that the wave function has the following form: $\Psi_n(y) = f_n(y) e^{-\beta y^2}$, where β is an adjustable parameter. The adjustable parameter β was introduced in order to improve its rate of convergence and should be positive; otherwise, the normalization condition of the wave function will be violated. It should yield the best convergence rate, i.e. the minimum number of iterations. Moreover, these solutions are valuable tools for checking and improving numerical methods introduced for solving complicated physical systems.

3. The harmonic oscillator potential

Considering the well-known case of the harmonic oscillator potential

$v(y) = b_2 y^2$, with b_2 positive, the necessary and sufficient conditions are:

$$G_n(y) = \frac{1}{A \exp(-h(y))} \frac{d^2\psi_n(y)}{dy^2} + (\varepsilon - b_2 y^2) f_n(y) = 0, \quad (7)$$

The node-less eigenfunction is given by

$\psi_0(y) = A f_0(y) \exp(-h(y))$ with $f_0(y) = 1$ and $h(y) = \sum_{p=1}^N a_{2p} y^{2p}$. Introducing this solution in Eq. (7) leads to the system:

$$N = 1, \quad (8)$$

$$a_2 = \frac{\sqrt{b_2}}{2}, \quad (9)$$

$$E_0 = \sqrt{b_2}, \quad (10)$$

However, the first eigenstate wave function can be written as $\psi_1(y) = A f_1(y) \exp(-h(y))$ with $f_1(y) = (y - y_{11})$, to be the single zero of the wave function and $h(y) = \sum_{p=1}^N a_{2p} y^{2p}$, Eq. (7) confirms that

$$N = 1, \quad (11)$$

$$a_2 = \frac{\sqrt{b_2}}{2}, \quad (12)$$

$$E_1 = 3\sqrt{b_2}, \quad (13)$$

$$y_{11} = 0, \quad (14)$$

But, the second eigenstate wave function $\psi_2(y) = A f_2(y) \exp(-h(y))$, where $f_2(y) = (y - y_{12})(y - y_{22})$, and the couple (y_{12}, y_{22}) constitutes the two zeros of the wave function; note that, they are opposite, i.e. $y_{12} = -y_{22}$. Eq. (7) offers this system of equations:

$$N = 1, \quad (15)$$

$$a_2 = \frac{\sqrt{b_2}}{2}, \quad (16)$$

$$E_2 = 5\sqrt{b_2}, \quad (17)$$

$$y_{12} = -y_{22} = \frac{1}{\sqrt{2b_2}}, \quad (18)$$

Nevertheless, The third eigenfunction is $\psi_3(y) = A f_3(y) \exp(-h(y))$ with $f_3(y) = (y - y_{13})(y - y_{23})(y - y_{33})$. Here y_{13}, y_{23} , and y_{33} are the wave function's zeros, let us take $y_{13} = 0$ and $(y_{23} = -y_{33})$. We obtain for this third eigenstate the following equations system:

$$N = 1, \quad (19)$$

$$a_2 = \frac{\sqrt{b_2}}{2}, \quad (20)$$

$$E_3 = 7\sqrt{b_2}, \quad (21)$$

$$y_{13} = 0, \quad (22)$$

$$y_{23} = -y_{33} = \frac{\sqrt{6}}{2b_2}, \quad (23)$$

To finish, the fourth eigenfunction is defined by $\psi_4(y) = A f_4(y) \exp(-h(y))$ with $f_4(y) = (y - y_{14})(y - y_{24})(y - y_{34})(y - y_{44})$ y_{14}, y_{24}, y_{34} and y_{44} are the wave function's zeros they are opposite two

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