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ORIGINAL ARTICLE

Computing multiple zeros using a class of quartically convergent methods

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Abstract Targeting a new multiple zero finder, in this paper, we suggest an efficient two-point class of methods, when the multiplicity of the root is known. The theoretical aspects are investigated and show that each member of the contributed class achieves fourth-order convergence by using three functional evaluations per full cycle. We also employ numerical examples to evaluate the accuracy of the proposed methods by comparison with other existing methods.

For functions with finitely many real roots in an interval, relatively little literature is known, while in applications, the users wish to find all the real zeros at the same time. Hence, the second aim of this paper will be presented by designing a fourth-order algorithm, based on the developed methods, to find all the real solutions of a nonlinear equation in an interval using the programming package MATHEMATICA 8.

1. Preliminaries

Many methods have been developed for solving nonlinear equations or systems using different methodology, see [2] and the references therein. On the other hand, solutions themselves, can be divided into the simple or the multiple cases. That is to say, a function might have finitely many zeros in an interval which some of them are simple while the other ones could be multiple.

A multiple zero is a root with multiplicity $m \geqslant 2$, also called a multiple point or repeated root. Clearly, working and developing on multiple roots of a nonlinear equation is not an easy task in numerical analysis. Herein, we develop iterative methods to find the multiple root x^* with multiplicity m of a nonlinear equation $f(x) = 0$, i.e., $f^{(i)}(x^*) = 0$, $i = 0, 1, \cdots, m-1$, and $f^{(m)}(x^*) \neq 0$. We will also discuss on finding simple zeros and also multiple zeros when the multiplicity of the roots are unknown.

When searching for multiple roots, there are some problems, which need special attention. The first one is that there is a neighborhood of x^*, here called the error ball, where the accurate computation of $f(x)$ is not possible because of

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computational (cancelation) errors, actually the errors are bigger than function values.

As a result, an in-appropriate iterative method returns an entirely erroneous root estimate such that it may break down. Even if having some methods to recognize error ball, we can only hope a result of lower accuracy [5]. A less severe-problem is slow convergence. Thus, it is more important if we can devise methods that ensure computation of the multiple roots at high precision.

Along with the main above questions, the most important problem will be remained in implementation. In fact, in applications, one might to find all the (real) critical points of a nonlinear function in a given interval, while such iterative methods are sensitive upon the initial points!

All of such needs and questions will be answered and solved in the following sections. It is assumed in the following that the derivatives of the function exist and are easily computable.

The remainder sections of this paper unfold the contents in what follows. In Section 2, a brief discussion on the existing methods in the literature will be given. In order to contribute, we first in Section 3 present a two-step method for finding simple zeros of nonlinear equations. Theoretical aspects support the quartical convergence. Then, we extend the scheme for approximating multiple zeros when the multiplicity of the zero is known. Section 4 dedicates to remind some of the ways of approximating the order of multiplicity. Next, in Section 5, we extend one of the methods from the suggested class of Section 3 to find multiple zeros when the multiplicity of the zeros is unknown. In Section 6, some tests will be given to show the numerical behavior of the attained iterative methods for finding multiple roots by comparison with the existing methods. Since all the obtained iteration functions up to this point, are locally fourth-order convergent, and in applications we need to have the convergence to be guaranteed alongside finding all the real zeros in an interval, in Section 7, we design an efficient *hybrid* algorithm to capture all the real solutions by applying our high-order optimal methods using the efficient programming package MATHEMATICA. Finally, Section 8 will be drawn the conclusion of this study with some outlines for future works.

2. Brief review

It is well-known that when the multiplicity m of the root is given then the modified Newton's method converges quadratically and can be defined in the following form $x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}$. According to the definition of efficiency index (defined in [27]), it has $2^{1/2} \approx 1.414$ as its index of efficiency. In fact, we considered that all function and derivative evaluations per computing step have the same computational cost. In order to provide better orders and efficiencies many developments have been given to the literature, see e.g. [22,26].

A third-order Chebyshev's method [27] for finding multiple roots is given by

$$x_{n+1} = x_n - \frac{m(3-m)}{2}\frac{f(x_n)}{f'(x_n)} - \frac{m^2}{2}\frac{f^2(x_n)f''(x_n)}{f'^3(x_n)}. \qquad (1)$$

A note on this scheme is that it needs the computation of the second derivative along the knowledge of multiplicity to be implemented, which is costly in many problems. Scheme (2) has the efficiency index 1.442.

Now, we recall an important finding in this topic regarding the optimal relation between the number of evaluations and the local order of convergence. The upper bound for order of multi-step methods was discussed in [12] by Kung and Traub, who conjectured that the order of convergence of any multipoint method without memory, consuming $n+1$, (functional) evaluations per iteration, cannot exceed the bound 2^n (called optimal order). This hypothesis has not been proved yet but it turned out that all existing methods constructed at present support it. Another interesting point is that this conjecture is valid for iterative methods, which are designed for simple zeros; or for multiple zeros with the known multiplicity. That is to say, if the multiplicity be unknown, then the conjecture is not anymore supported.

In case of multiple root solvers which are optimal, we can name the following schemes. Sharifi et al. in [18] proposed the following optimal fourth-order method

$$\begin{cases} y_n = x_n - \frac{2m}{m+2}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n + \left(\frac{1}{4}m(m^2+2m-4)u_n - \frac{1}{4}m(m+2)^2p^m v_n\right) \\ \times\left(1 + \frac{m^4}{8(m+2)p^{2m}}(w_n)^2 - \frac{69}{64}(w_n)^3 + v_n^4\right). \end{cases} \qquad (2)$$

wherein $u_n = \frac{f(x_n)}{f'(x_n)}$, $v_n = \frac{f(x_n)}{f'(y_n)}$, $w_n = \frac{f'(y_n)}{f'(x_n)} - p^{m-1}$ and $p = \frac{m}{m+2}$.

Sharma and Sharma in [19] suggested the following quartically technique

$$\begin{cases} y_n = x_n - \frac{2m}{m+2}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \frac{m}{8}\left((m^3-4m+8) - (m+2)^2\left(\frac{m}{m+2}\right)^m\frac{f'(x_n)}{f'(y_n)}(2(m-1)\right. \\ \left.-(m+2)\left(\frac{m}{m+2}\right)^m\frac{f'(x_n)}{f'(y_n)})\right)\frac{f(x_n)}{f'(x_n)}, \end{cases} \qquad (3)$$

and Zhou et al. in [30] presented the following scheme with the same order as (2), (3)

$$\begin{cases} y_n = x_n - \frac{2m}{m+2}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \frac{m}{8}\left((m^3+6m^2+8m+8) + m^3\left(\frac{m+2}{m}\right)^{2m}\left(\frac{f'(y_n)}{f'(x_n)}\right)^2\right. \\ \left.-2m^2(m+3)\left(\frac{m+2}{m}\right)^m\frac{f'(y_n)}{f'(x_n)}\right)\frac{f(x_n)}{f'(x_n)}. \end{cases} \qquad (4)$$

The techniques (2)–(4) approximate the multiple roots, when the multiplicity of the root is available by consuming three (functional) evaluations. They also possess the optimal order four and the optimal efficiency index 1.587 in the sense of Kung–Traub.

For further related developments and applications, see [16,20,21,23–25], where some other aspects of nonlinear equation solving by iterative methods have been discussed.

Inspired by these new optimal developments and also by the use of weight function, we present in the next section, a general technique for solving nonlinear scalar equations.

3. Construction of the new technique

In this section, we derive a new technique of two-point methods of order four, requiring three (functional) evaluations per