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ORIGINAL ARTICLE

Navier–Stokes flow in converging–diverging distensible tubes

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Abstract We use a method based on the lubrication approximation in conjunction with a residual-based mass-continuity iterative solution scheme to compute the flow rate and pressure field in distensible converging–diverging tubes for Navier–Stokes fluids. We employ an analytical formula derived from a one-dimensional version of the Navier–Stokes equations to describe the underlying flow model that provides the residual function. This formula correlates the flow rate to the boundary pressures in straight cylindrical elastic tubes with constant-radius. We validate our findings by the convergence toward a final solution with fine discretization as well as by comparison to the Poiseuille-type flow in its convergence toward analytic solutions found earlier in rigid converging–diverging tubes. We also tested the method on limiting special cases of cylindrical elastic tubes with constant-radius where the numerical solutions converged to the expected analytical solutions. The distensible model has also been endorsed by its convergence toward the rigid Poiseuille-type model with increasing the tube wall stiffness. Lubrication-based one-dimensional finite element method was also used for verification. In this investigation five converging–diverging geometries are used for demonstration, validation and as prototypes for modeling converging–diverging geometries in general.

1. Introduction

The flow of fluids in converging–diverging tubes has many scientific, technological and medical applications such as transportation in porous media, filtration processes, polymer processing, and pathological stenoses and aneurysms [1–13]. There are many studies about the flow in converging–diverging rigid conduits [14–21] and distensible conduits with fixed cross sections [22–28] separately as well as many other different geometries and fluid and conduit mechanical properties [29–31]. There is also a considerable number of studies on the flow in converging–diverging distensible conduits; although large part of which is related to medical applications such as stenosis and blood flow modeling [32–42].

Several methods have been used in the past for investigating and modeling the flow in distensible converging–diverging geometries; the majority are based on the numerical discretization methods such as finite element and spectral methods although other approaches such as stochastic techniques have also been employed. However, due to the huge difficulties associating this subject which combines tube wall deformability with convergence–divergence non-linearities, most of these

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Nomenclature

α	correction factor for axial momentum flux	p	pressure
β	stiffness coefficient in the pressure-area relation	$\mathbf{p}$	pressure vector
κ	viscosity friction coefficient	p_i	inlet pressure
μ	fluid dynamic viscosity	p_o	outlet pressure
ν	fluid kinematic viscosity	Δp	pressure drop
ρ	fluid mass density	$\mathbf{\Delta p}$	pressure perturbation vector
ς	Poisson's ratio of tube wall	Q	volumetric flow rate
A	tube cross-sectional area at actual pressure	Q_a	analytic flow rate for rigid tube
A_{in}	tube cross-sectional area at inlet	Q_e	numeric flow rate for elastic tube
A_o	tube cross-sectional area at reference pressure	Q_r	numeric flow rate for rigid tube
A_{ou}	tube cross-sectional area at outlet	$\mathbf{r}$	residual vector
E	Young's elastic modulus of the tube wall	R	tube radius
f	flow continuity residual function	R_{max}	maximum unstressed tube radius
h_o	tube wall thickness at reference pressure	R_{min}	minimum unstressed tube radius
$\mathbf{J}$	Jacobian matrix	t	time
L	tube length	x	tube axial coordinate (inlet at $x = 0$ and outlet at $x = L$)
N	number of discretized tube nodes		

studies are based on substantial approximations and modeling compromises. Moreover, they are usually based on very complex mathematical and computational infrastructures which are not only difficult to implement and use but also difficult to verify and validate. Also, some of these methods, such as stochastic techniques, are computationally demanding and hence they may be prohibitive in some cases. Therefore, simple, reliable and computationally low cost techniques are highly desirable where analytical solutions are not available due to excessive difficulties or even impossibility of obtaining such solutions which is the case in most circumstances.

In this paper we propose the use of the lubrication approximation with a residual-based non-linear solution scheme in association with an analytical expression for the flow of Navier–Stokes fluids in straight cylindrical elastic tubes with fixed radius to obtain the flow rate and pressure field in a number of cylindrically-symmetric converging–diverging geometries with elastic wall mechanical properties. The proposed method combines simplicity, robustness and ease of implementation. Moreover, it produces solutions which are very close to any targeted analytical solutions as the convergence behavior in the investigated special cases reveals.

Although the proposed method is related to a single distensible tube, it can also be extended to a network of interconnected distensible tubes with partially or totally converging–diverging conduits by integrating these conduits into the network and giving them a special treatment based on the proposed method. This approach, can be utilized for example in modeling stenoses and other types of flow conduits with irregular geometries as part of fluid flow networks in the hemodynamic and hemorheologic studies and in the filtration investigations.

The method also has a wider validity domain than what may be thought initially with regard to the deformability characteristics. Despite the fact that in this paper we use a single analytical expression correlating the flow rate to the boundary pressures for a distensible tube with elastic mechanical properties, the method can be well adapted to other types of mechanical characteristics, such as tubes with viscoelastic wall rheology, where different pressure-area constitutive relations do apply. In fact there is no need even to have an analytical solution for the underlying flow model that provides the basic flow characterization for the discretized elements of the converging–diverging geometries in the lubrication approximation. What is actually needed is only a well defined flow relation: analytical, or empirical, or even numerical [43] as long as it is viable to find the flow in the discretized elements of the lubrication ensemble using such a relation to correlate the flow rate to the boundary pressures.

There is also no need for the geometry to be of a fixed or regular shape as long as a characteristic flow can be obtained on the discretized elements, and hence the method can be applied not only to axi-symmetric geometries with constant-shape and varying cross-sectional area in the flow direction but can also be extended to non-symmetric geometries with irregular and varying shape along the flow direction if the flow in the deformable discretized elements can be characterized by a well-defined flow relation. The method can as well be applied to non-straight flow conduits with and without regular or varying cross-sectional shapes such as bending compliant pipes.

2. Method

The flow of Navier–Stokes fluids in a cylindrical tube with a cross-sectional area A and length L assuming a slip-free incompressible laminar axi-symmetric flow with negligible gravitational body forces and fixed velocity profile is described by the following one-dimensional system of mass continuity and linear momentum conservation principles

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \quad t \geqslant 0, \;\; x \in [0, L] \tag{1}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\alpha Q^2}{A}\right) + \frac{A}{\rho}\frac{\partial p}{\partial x} + \kappa\frac{Q}{A} = 0 \quad t \geqslant 0, \;\; x \in [0, L] \tag{2}$$