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ORIGINAL ARTICLE

A new fractional modeling arising in engineering sciences and its analytical approximate solution

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Abstract The aim of this article is to introduce a new approximate method, namely homotopy perturbation transform method (HPTM) which is a combination of homotopy perturbation method (HPM) and Laplace transform method (LTM) to provide an analytical approximate solution to time-fractional Cauchy-reaction diffusion equation. Reaction diffusion equation is widely used as models for spatial effects in ecology, biology and engineering sciences. A good agreement between the obtained solution and some well-known results has been demonstrated. The numerical solutions obtained by proposed method indicate that the approach is easy to implement and accurate. Some numerical illustrations are given. These results reveal that the proposed method is very effective and simple to perform for engineering sciences problems.

1. Introduction

The subject of factional calculus (integral and derivatives of any arbitrary real or complex) was planted over 300 year ago. The theory of derivative and integrals of non-integer order goes back to Liouville, Leibnitz, Grunwald, Reimann and Letnikov. In the recent years, fractional calculus has played a very significant role in many areas in fluid flow, mechanics, viscoelasticity, biology, physics, science and engineering, and other applications [1–4]. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Half-order derivatives and integrals proved to be more useful for the formulation of certain electrochemical problems than the classical models [3–6].

The Cauchy-reaction diffusion equations describe a wide variety of nonlinear systems in physics, chemistry, ecology, biology and engineering [7–10]. Recently, Yildirim [11] have applied to obtain the solutions of the Cauchy-reaction diffusion equations by using homotopy perturbation method. The main aim of this article presents approximate analytical solutions of fractional model of Cauchy-reaction diffusion equations with fractional time derivative α $(0 < \alpha \leqslant 1)$ in the form of a rapidly convergent series with easily computable components by using new homotopy perturbation transform method. Using the initial condition, the approximate analyti-

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cal expressions for different fractional Brownian motions and also for standard motion are obtained. The elegance of this article can be attributed to the simplistic approach in seeking the approximate analytical solution of the time-fractional Cauchy-reaction diffusion equations.

Thus seeking solutions of nonlinear fractional ordinary and partial differential equations is still a significant task. Except in a limited numbers of these equations, we have difficulty to finding their analytical as well as approximate solutions. Therefore, there have been attempts to develop the new methods for obtaining analytical and approximate solutions of nonlinear fractional ordinary and partial differential equations. Recently, several methods have drawn special attention such as Adomian decomposition method [12–14], Variational iteration method [15,16], Homotopy analysis method [17–20], Differential transform method [21,22], Wavelet methods [23,24], and Homotopy perturbation method [25–36].

The main aim of this article is to illustrate how the Laplace transform can be used to find analytical and approximate solutions of the linear and nonlinear fractional differential equations by manipulating the homotopy perturbation method. The homotopy perturbation method introduced and applied by He [25–29]. Recently, many researchers [30–36] have obtained the series solution of the fractional differential equations and integral equation by using HPM. The proposed method is coupling of the Laplace transformation, the homotopy perturbation method and He's polynomials mainly due to Ghorbani [37,38]. In the recent years, many authors have paid attention to study the solutions of linear and nonlinear partial differential equations by using various methods with combined the Laplace transform. Among these are the Laplace decomposition methods [39,40], homotopy perturbation transform method [41–45].

Definition1.1. The Laplace transform of function $f(t)$ is defined by

$$F(s) = L[f(t)] = \int_0^\infty e^{-st} f(t)dt. \tag{1.1}$$

Definition1.2. The Laplace transform $L[f(t)]$ of the Riemann–Liouville fractional integral is defined as [2]

$$L[I_t^\alpha f(t)] = s^{-\alpha} F(s). \tag{1.2}$$

Definition1.3. The Laplace transform $L[f(t)]$ of the Caputo fractional derivative is defined as [2]

$$L\left[D_t^{n\alpha} f(t)\right] = s^{n\alpha} F(s) - \sum_{k=0}^{n-1} s^{(n\alpha-k-1)} f^{(k)}(0), \quad n-1 < n\alpha \leqslant n. \tag{1.3}$$

Definition1.4. The Mittag–Leffler function $E_\alpha(z)$ with $\alpha > 0$ is defined by the following series representation, valid in the whole complex plane [46]:

$$E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + 1)}. \tag{1.4}$$

2. Basic idea of newly fractional homotopy perturbation transform method

In order to elucidate the solution procedure of the fractional homotopy perturbation transform method, we consider the following nonlinear fractional differential equations:

$$D_t^{n\alpha} u(x,t) + R[x]u(x,t) + N[x]u(x,t) = q(x,t), \quad t > 0, \\ \in R, \quad n-1 < n\alpha \leqslant n, u(x,0) = h(x), \tag{2.1}$$

where $D_t^{n\alpha} = \frac{\partial^{n\alpha}}{\partial t^{n\alpha}}$, $R[x]$ is the linear operator in x, $N[x]$ is the general nonlinear operator in x, and $q(x,t)$ is continuous function. Now, the methodology consists of applying Laplace transform on both sides of Eq. (2.1), we get

$$L[D_t^{n\alpha} u(x,t)] + L[R[x]u(x,t) + N[x]u(x,t)] = L[q(x,t)]. \tag{2.2}$$

Now, using the differentiation property of the Laplace transform, we have

$$L[u(x,t)] = s^{-1}h(x) + s^{-n\alpha} L[q(x,t)] - s^{-n\alpha} L[R[x]u(x,t) \\ + N[x]u(x,t)]. \tag{2.3}$$

Operating the inverse Laplace transform on both sides in Eq. (2.3), we get

$$u(x,t) = G(x,t) - L^{-1}(s^{-n\alpha} L[R[x]u(x,t) + N[x]u(x,t)]), \tag{2.4}$$

where $G(x, t)$, represents the term arising from the source term and the prescribed initial conditions. Now, applying the classical perturbation technique, we can assume that the solution can be expressed as a power series in p as given below

$$u(x,t) = \sum_{n=0}^\infty p^n u_n(x,t), \tag{2.5}$$

where the homotopy parameter p is considered as a small parameter ($p \in [0,1]$). The nonlinear term can be decomposed as

$$Nu(x,t) = \sum_{n=0}^\infty p^n H_n(u), \tag{2.6}$$

where H_n are He's polynomials of u_0, u_1, u_2, ..., u_n and it can be calculated by the following formula

$$H_n(u_0,u_1,u_2,\ldots,u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^\infty p^i u_i\right)\right]_{p=0}, \quad n = 0,1,2,\ldots$$

Substituting Eqs. (2.5) and (2.6) in Eq. (2.4) and using HPM [25–29], we get

$$\sum_{n=0}^\infty p^n u_n(x,t) = G(x,t) - p\left(L^{-1}\left[s^{-n\alpha} L\left[R\sum_{n=0}^\infty p^n u_n(x,t) + \sum_{n=0}^\infty p^n H_n(u)\right]\right]\right). \tag{2.7}$$

This is coupling of the Laplace transform and homotopy perturbation method by using He's polynomials. Now, equating the coefficient of corresponding power of p on both sides, the following approximations are obtained as

$$p^0 : u_0(x,t) = G(x,t),$$
$$p^1 : u_1(x,t) = L^{-1}(s^{-n\alpha} L[R[x]u_0(x,t) + H_0(u)]),$$
$$p^2 : u_2(x,t) = L^{-1}(s^{-n\alpha} L[R[x]u_1(x,t) + H_1(u)]),$$
$$p^3 : u_3(x,t) = L^{-1}(s^{-n\alpha} L[R[x]u_2(x,t) + H_2(u)]).$$

Proceeding in this same manner, the rest of the components $u_n(x,t)$ for all n > 3 can be completely obtained and the series solution is thus entirely determined.