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ORIGINAL ARTICLE

# The energy balance to nonlinear oscillations via Jacobi collocation method

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**Abstract** This study develops the energy balance based on Jacobi collocation method for accurate prediction of conservative nonlinear oscillator models with a single collocation point. The node points are taken as the roots of Jacobi orthogonal polynomials. Several examples are included to demonstrate the applicability and accuracy of the proposed algorithm, and some comparisons are made with the existing results. The method is suitable and the approximate frequencies are valid for small as well as large amplitudes of oscillation. Excellent agreement with exact ones is presented for the first order approximation.



## 1. Introduction

The accurate prediction of nonlinear oscillator systems is of significant importance in many areas of physics, mechanics and structural dynamics. Hence, a large variety of analytical techniques have been widely considered [1–50]. For instance, Belendez [14] used homotopy perturbation method to obtain the periodic solutions of a conservative nonlinear oscillator. Cveticanin et al. [4] exerted the energy balance method including the Petrov–Galerkin approach for a generalized oscillator with strong nonlinear terms, He [33] applied the variational iteration method to solve nonlinear differential equations with convolution product nonlinearities, Pirbodaghi et al. [34] investigated non-linear vibrational behavior of Euler–Bernoulli beams subjected to axial loads using the homotopy analysis method, Sedighi et al. [35] promoted the application of Max–Min approach to the governing equation of transversely vibrating quintic nonlinear beams, Khan et al. [36] studied a generalized conservative nonlinear equation by means of frequency–amplitude formulation, and Lai et al. [31] utilized Newton–harmonic balancing approach for accurate solutions to nonlinear cubic–quintic Duffing oscillators.

Moreover, researchers successfully applied the parameter expansion method (PEM) to various engineering problems such as; the dynamic instability of functionally graded nanobridges in the presence of Casimir attraction and electric filed actuation [43], lateral vibrations of a nonlinear beam resting on an elastic foundation [44], nonlinear vibrations of a cantilever beam with a discontinuous dead zone nonlinearity [45] and the dynamical analysis of beam vibrations in the presence of the preload discontinuity [46]. As well as Sedighi and Shirazi [47] analyzed nonlinear dynamical behavior of a railway bogie using the Bogoliubov Averaging method. They also investigated the vibrational behavior of the Euler–Bernoulli beam exposed to saturated nonlinear boundary

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condition by Hamiltonian approach [48]. Furthermore, Sedighi and Daneshmand [49] employed the homotopy perturbation method with an auxiliary term for nonlinear transversely vibrating beams having odd and even nonlinearities. In addition, they examined the iteration perturbation method (IPM) for static and dynamic pull-in behavior of carbon nanotube (CNT) probes near graphite sheets [50].

The energy balance method, was first propounded by He [1]. Applicability of this reliable and straightforward technique has been evaluated in dealing with nonlinear oscillator systems. Among the nonlinear problems one may refer to nonlinear oscillations of a punctual charge in the electric field of a charged ring [37], strongly nonlinear generalized Duffing oscillators [38], the Helmholtz–Duffing equation [39], the Duffing–relativistic oscillator [40], free vibrations of an autonomous conservative oscillator with inertia and static type fifth-order non-linearities [41] and nonlinear dynamics of a mass attached to a stretched wire [42].

In this study, accuracy of the method is improved via Jacobi collocation method. The collocation method belongs to the family of weighted residual methods [51]. Indeed, the weighting function in the collocation method is the Dirac delta function and the residual at a given number of individual points is reduced to zero. Choosing the location of the points, causes the accurate results achieved using roots of Jacobi polynomials [52].

The present study intends to extend the application of Jacobi polynomials to obtain angular frequencies of nonlinear oscillators by the collocation method. Owing to optimal selection of orthogonal polynomials, the value of the parameters of the Jacobi polynomial, $\alpha$ and $\beta$, can be used to tune the location of the roots within the domain.

Some examples reveal that even the first order approximations lead to high accuracy for all values of oscillation amplitude. The presented results display that the technique is very effective and convenient for conservative nonlinear oscillators for which the restoring force has different forms.

The rest of the manuscript is organized as follows. Outlines of the Jacobi orthogonal polynomials and the Energy balance method are provided in Sections 2 and 3, respectively. The examples are presented in Section 4. Section 5 ends this study with a brief conclusion.

## 2. Jacobi orthogonal polynomials

This section gives some definitions and properties of Jacobi polynomials, $P_n^{(\alpha,\beta)}(x)$. The Jacobi polynomials are defined as follows:

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} F\left(-n, 1+\alpha+\beta+n; \alpha+1; \frac{1-x}{2}\right) \tag{1}$$

where $F$ and $(\lambda)_n$ are the hypergeometric function and the Pochhammer symbol, respectively and defined as follows:

$$F(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{z^n}{n!} \tag{2}$$

$$(\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1) = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}. \tag{3}$$

The Jacobi polynomials with general parameters $(\alpha, \beta > -1)$ satisfy the orthogonality relations as follows:

$$\int_{-1}^{+1} (1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) dx = C_n \delta_{nm} \tag{4}$$

where $\delta_{nm}$ is the Kronecker function, and

$$C_n = \frac{2^{\alpha+\beta+1}(n+\alpha+1)\Gamma(n+\beta+1)}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}. \tag{5}$$

The Jacobi polynomials are the eigenfunctions of the following singular Sturm–Liouville equation:

$$(1-x^2)y'' + [\beta - \alpha - (\alpha+\beta+2)x]y' + n(n+\alpha+\beta+1)y = 0. \tag{6}$$

An equivalent definition for the Jacobi polynomials is given by Rodrigues formula as follows:

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right]. \tag{7}$$

Besides,

$$\frac{d}{dx}\left[P_n^{(\alpha,\beta)}(x)\right] = \frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x), \tag{8}$$

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x). \tag{9}$$

## 3. Energy balance method

Consider a general nonlinear oscillator as follows:

$$u'' + f(u) = 0, \quad u(0) = A, \quad u'(0) = 0. \tag{10}$$

The variational principle of Eq. (10) can be easily obtained as follows:

$$J(u) = \int_0^t \left(-\frac{1}{2}u'^2 + F(u)\right) dt \tag{11}$$

where $F(u) = \int f(u)du$.

Its Hamiltonian can be written in this form:

$$H = \frac{1}{2}u'^2 + F(u) = F(A), \tag{12}$$

or

$$\frac{1}{2}u'^2 + F(u) - F(A) = 0. \tag{13}$$

The following trial function is utilized to determine the angular frequency:

$$u = A\cos(\omega t). \tag{14}$$

Substituting (14) into (13), the following residual equation is obtained:

$$R(t) = \frac{1}{2}(A\omega)^2 \sin^2(\omega t) + F(A\cos(\omega t)) - F(A). \tag{15}$$

Since Eq. (14) is only an approximation to the exact solution, Eq. (15) cannot be made zero everywhere. Collocation at $\omega t = \kappa\pi$ gives

$$\omega_a = (A\sin(\kappa\pi))^{-1}(2(F(A) - F(A\cos(\kappa\pi))))^{\frac{1}{2}}. \tag{16}$$