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ORIGINAL ARTICLE

Why golden rectangle is used so often by architects: A mathematical approach

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Abstract It is often found in the literature that many researchers have studied or documented the use of golden rectangle or Fibonacci rectangle in architectural design. In this way, a lot of well-known architects in the history, knowingly or unknowingly, have employed either the golden rectangle or the Fibonacci rectangle in their works. Using some mathematical tools, this paper tried to approach one of the properties of the golden rectangle (or the Fibonacci rectangle) and its significance to architectural design, which could lead to state one hypothesis about why architects have used them so often.

This work begins with an algorithm which constructs a Fibonacci rectangle and a golden rectangle. Then adjacency among the squares (which are arranged inside them) is defined, by considering each square as a room or an architectural space. At the end, using some tools of the graph theory, it has been proved that they are one of the best arrangements of squares (or rectangles) inside a rectangle, from the point of view of connectivity.

1. Introduction

1.1. The golden ratio

The *golden number* is, in a sense, the most natural real number, since it can be written as:

$\varphi = [\overline{1}]$,

without reference to any numbering system. This is standard abbreviation for the continued fraction expansion

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

From this, it can be seen that $\varphi - 1 = \frac{1}{\varphi}$, so that φ is also equal to $\frac{1+\sqrt{5}}{2}$. The golden number is also the limit of the sequence of *convergents*

$$(p_n/q_n)_{n=0}^{\infty} = \left(\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \ldots\right),$$

which involves the successive Fibonacci numbers. This is not just one more occurrence of the Fibonacci numbers; in fact, a classical result (see Hardy & Wright [1], Chapter 10) is that the sequence of convergents is a sequence of *best approximations* to φ by rational numbers, in a very specific sense. So, instead of saying that $\varphi = 1.61803\ldots$, which is meaningful

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only in the decimal system, it is better to consider φ as a sequence of convergents

$$\varphi = \left(\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \ldots\right).$$

1.2. Relation between the golden rectangle and the Fibonacci rectangle

The *golden rectangle* is a rectangle whose ratio of width over height is equal to φ with a geometric property as follows:

One can remove a square with side length one from a rectangle of sides $1 \times \varphi$ and obtain a new rectangle, with sides $\frac{1}{\varphi} \times 1$, which is similar to the original one. Hence, the construction can be repeated (see Walser [2], Chapter 3). The golden rectangle and logarithmic spiral are shown in Fig. 1. The *logarithmic spiral* is centered at the point O which is the intersection of the two diagonals BD and CE_1. Its radius r is reduced by a factor φ each time the angle θ is decreased by $\pi/2$.

The golden rectangle is considered as one of the shape for representing φ in two dimensions (refer [3]). Because of this, φ and golden rectangle have same properties as well as the most visually pleasing constructions.

In a *Fibonacci sequence*, each of its term is obtained from the sum of the two preceding terms i.e. $F_{n+1} = F_n + F_{n-1}$ for $n > 1$ where $F_0 = F_1 = 1$. Here each F_n is called Fibonacci number. A *Fibonacci rectangle* is a rectangle with side lengths x and y such that either x/y or y/x is equal to F_{n+1}/F_n for some non-negative integer n. Naturally, one can construct such a rectangle by successively introducing squares of side lengths $F_0, F_1, F_2, \ldots$ as shown in Fig. 5. It can be easily seen that the ratio of two successive Fibonacci numbers (F_{n+1}/F_n) approaches φ.

If only the arrangement of squares is considered, the two Figures, golden rectangle and Fibonacci rectangle, look similar, whereas, geometrically it is far more convenient to use the Fibonacci rectangle in comparison with the golden rectangle, because, it is comparatively easy to draw a rectangle with integer dimensions (say 5×8) than a rectangle having rational or irrational dimensions respectively.

1.3. Golden rectangle, Fibonacci rectangle and architecture

The φ or golden rectangle has been found in the natural world through human proportions and through growth patterns of many living plants, animals, and insects. Basically, it has been

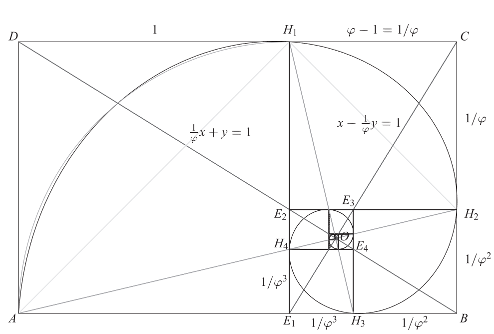

Figure 1 The golden rectangle and logarithmic spiral.

always considered that φ is the most pleasing proportion to human eyes [4,5]. The presence of φ in the design of the Pyramids represents that the Egyptians were aware of the number. A Greek sculptor and mathematician, Phidias (490–430 BC), was first to study and apply Phi, to the design of sculptures for the Parthenon (example of Doric architecture, the main temple of the goddess Athena) [5,6].

Around 1200 AD, Leonardo Fibonacci (1170–1250 AD), an Italian born mathematician found φ in a numerical series (known as Fibonacci series) and named it *divine proportion*, due to which, Fibonacci series can be used to construct the golden rectangle [3]. The design of Notre Dame in Paris, which was built in between 1163 and 1250, appears to have golden rectangle in number of its key proportions. Likewise, the Renaissance artists used the golden rectangle in their various paintings and sculptures to achieve balance and esthetic beauty [7].

Also, φ is been favorite to many key architects in history, such as, Palladio, Le Corbusier, Pacioli, and Leonardo Da Vinci. Palladio's Villa La Rotonda is designed using φ, see Figs. 12–14 (for details of Palladio's work, refer [8], *I Quattro Libri dell'Architettura*). Le Corbusier himself wrote *Modulor I and II* (refer [9]) where for instance he displayed the composition and drawing of his 'Modulor' figure by using the golden ratio. As a sculptor and applied mathematician, George Hart [10] delves the work of Pacioli and Leonardo Da Vinci given in the book *De Divina Proportione*.

In addition, many recent publications discussed the golden ratio and architectural designs that exist at the different moments of history. In 1986, Burckhardt [11] studied the presence of golden rectangle in a house of 1871 in Basel. In 2000, Mark Reynolds [12] presented the use of the golden section in generating the geometry of Pazzi Chapel of Santa Croce in Florence. In 2013, Fernández-Llebrez and Fran [13] discussed the presence of the golden section and the Fibonacci sequence in the compositional scheme of the Roman Catholic Church Pastoor Van Ars, built by Aldo van Eyck in The Hague in 1968.

It is hard to find many publications showing the presence of Fibonacci sequence in the geometrical composition of architectural design; two of them are mentioned as follows:

Bartoli [14] discussed the case of Palazzo della Signoria, where the Fibonacci rectangle has been found (see Figs. 15 and 16). In the same way, Park and Lee [15] published the underlying design of the Braxton-Shore house by Rudolph Schindler, which is based on the Fibonacci sequence.

From the work of many authors and researchers, it can be easily seen that many buildings and architectural designs (from ancient to contemporary time) have been developed using the golden rectangle or the Fibonacci rectangle, but it is difficult to find a mathematical or logical reason behind it. That is why, this paper tried to provide a possible mathematical explanation to this question in terms of adjacency among the rooms of an architectural design.

2. Rectangular arrangements and connectivity

2.1. Rectangular arrangements

A *rectangular arrangement* is defined as an arrangement of sub-rectangles inside a bigger rectangle where: