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ORIGINAL ARTICLE

A new analytical technique based on harmonic balance method to determine approximate periods for *Duffing-harmonic* oscillator

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Abstract The *Duffing-harmonic* oscillator is a common model for nonlinear phenomena in science and engineering. In this paper, a new analytical technique has been presented to determine approximate periods of a strongly nonlinear *Duffing-harmonic* oscillator. Generally, a set of difficult nonlinear algebraic equations appear when harmonic balance method is imposed. The power series solutions of these equations are invalid. The proposed idea avoids this limitation and the necessity of numerically solving such nonlinear algebraic equations with very complex nonlinearities. In this technique, different parameters for the same nonlinear problems are found, for which the power series solution yields desired results. Besides a suitable truncation formula is found in which the solution measures better results than existing solutions. It is remarkable that this procedure is simple and takes less computational effort for determining second and higher order periods of oscillation for such nonlinear problems and shows a good agreement compared with the exact ones.

1. Introduction

Many fundamental laws of physics, mechanical, chemical, biochemical, biological, and engineering problems appear mathematically in the form of differential equations which are linear or nonlinear, autonomous or non-autonomous. Nonlinear oscillations are important fact in physical science, mechanical structures and other engineering problems. Practically, all differential equations involving engineering and physical phenomena are nonlinear. The methods of solution procedures of linear differential equations are comparatively easy and well established. On the contrary, the solution procedures of nonlinear differential equations (NDEs) are less available and, in general, linear approximations are frequently used. The method of small oscillations is a well-known example of the linearization of problems, which are essentially nonlinear. With the discovery of numerous phenomena of self-excitation of a strongly nonlinear *Duffing-harmonic* oscillator, a rigid rod rocking back and forth on a circular surface without slipping

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and in many cases of nonlinear mechanical vibrations of special types, the methods of small oscillations become inadequate for their analytical treatment. There exists an important difference between the phenomena which oscillate in steady state and the phenomena governed by the linear differential equations with constant coefficients, e.g., oscillation of a pendulum with small amplitude, in that the amplitude of the ultimate stable oscillation seems to be entirely independent of initial conditions, whereas in oscillations governed by the linear differential equations with constant coefficients, it depends upon the initial conditions.

Recently, nonlinear process is one of the biggest challenges and is not easy to control because the nonlinear characteristics of the system dramatically change the solution due to some slight changes of valid parameters including time. Thus, the issue becomes more complicated and hence, needs ultimate solution. Therefore, the studies of approximate solutions of NDEs play a crucial role in understanding the internal mechanism of nonlinear phenomena. Advanced nonlinear techniques are significant to solve inherent nonlinear problems; particularly those are involved in differential equations, dynamical systems and related areas. Nowadays, both mathematicians and physicists have made significant improvement in finding a new mathematical tool related to NDEs and dynamical systems whose understanding will rely not only on analytic techniques but also on numerical and asymptotic methods. They establish many effective and powerful methods to handle the NDEs.

The study of given nonlinear problem is of crucial importance not only in all areas of physics but also in engineering and other disciplines, since most phenomena in our world are essentially nonlinear and are described by NDEs. Moreover, obtaining exact solutions for these problems has many difficulties. It is difficult to solve nonlinear problems and in general, it is often more difficult to get an analytic approximation than a numerical one for a given nonlinear problem. To overcome the shortcomings, many new analytical methods have been proposed nowadays. One of the widely used techniques is perturbation and another is asymptotic method [1–4], whereby the solution is expanded in powers of a small parameter. For strong nonlinearity, perturbation techniques do not provide expected results. However, for the nonlinear conservative systems, generalizations of some of the standard perturbation techniques overcome this limitation. In particular, generalization of Lindstedt–Poincare method and He's homotopy perturbation method yields desired results for strongly nonlinear oscillators [5–11]. Several authors used many other powerful various analytical methods in the field of approximate solutions especially for strongly nonlinear oscillators such as Max–Min Approach (MMA), Parameter Expansion Method (PEM), Variational Iteration Method (VIM), Amplitude Frequency Formulation (AFF), Energy Balance Method (EBM) and Enhanced Cubication Method (ECM) for solving NDEs arising in dynamical systems [12–17].

The harmonic balance method (HBM) is another technique for solving strongly nonlinear oscillators [18–33]. Usually, a set of difficult nonlinear algebraic equations appear when HBM is formulated. The higher order approximate periods (mainly third-order approximation) have been obtained for strongly nonlinear *Duffing-harmonic* oscillator. Comparison of the periods obtained in this study with exact periods shows that this method is effective and convenient for solving these analytical results. However, a suitable truncation of these algebraic equations takes the solution very close to the next higher-order approximation and also it saves a lot of calculations. This is the main advantage of the technique presented in this article.

2. Solution approaches

Let us consider a nonlinear differential equation

$$\ddot{x} + \omega_0^2 x = -\varepsilon f(x, \dot{x}), \quad x(0) = a_0, \ \dot{x}(0) = 0, \tag{1}$$

where $f(x, \dot{x})$ is a nonlinear function such that $f(-x, -\dot{x}) = -f(x, \dot{x}), \omega_0 \geqslant 0$ and ε is a constant.

A period solution of Eq. (1) can be written as

$$x = a_0(\rho\cos(\omega t) + u\cos(3\omega t) + v\cos(5\omega t) + w\cos(7\omega t) + z\cos(9\omega t)\cdots), \tag{2}$$

where a_0, ρ and ω are constants. If $\rho = 1 - u - v - \cdots$ and the initial phase $(\omega t)_0 = 0$, solution Eq. (2) easily satisfies the given initial conditions.

Substituting Eq. (2) into Eq. (1) and then expanding $f(x, \dot{x})$ in a Fourier series, Eq. (1) reduces to an algebraic identity as follows:

$$a_0[\rho(\omega_0^2 - \omega^2)\cos(\omega t) + u(\omega_0^2 - 9\omega^2)\cos(3\omega t) + \cdots] = -\varepsilon[F_1(a_0, u, \cdots)\cos(\omega t) + F_3(a_0, u, \cdots)\cos(3\omega t) + \cdots] \tag{3}$$

Now comparing the coefficients of equal harmonics of Eq. (3), we obtain the following nonlinear algebraic equations:

$$\rho(\omega_0^2 - \omega^2) = -\varepsilon F_1, \quad u(\omega_0^2 - 9\omega^2) = -\varepsilon F_3, \quad v(\omega_0^2 - 25\omega^2) = -\varepsilon F_5, \cdots \tag{4}$$

Applying the first equation in Eq. (4), ω^2 is eliminated from all the remaining equations of Eq. (4). Thus Eq. (4) takes the following form:

$$\rho\omega^2 = \rho\omega_0^2 + \varepsilon F_1, \quad 8\omega_0^2 u\rho = \varepsilon(\rho F_3 - 9uF_1), \quad 24\omega_0^2 v\rho = \varepsilon(\rho F_5 - 25vF_1), \ \cdots \tag{5}$$

Substituting $\rho = 1 - u - v - \cdots$, and simplifying, second-, third- equations in Eq. (5) reduce to

$$u = G_1(\omega_0, \varepsilon, a_0, u, v, \cdots, \lambda_0), \quad v = G_2(\omega_0, \varepsilon, a_0, u, v, \cdots, \lambda_0), \ \cdots, \tag{6}$$

where $G_1, G_2, \cdots$ exclude respectively the linear terms of $u, v, \cdots$.

For any values of ω_0, ε and a_0, there exists a parameter $\lambda_0(\omega_0, \varepsilon, a_0) \ll 1$, such that $u, v, \cdots$ are expandable into the power series in terms of λ_0 as

$$u = U_1\lambda_0 + U_2\lambda_0^2 + \cdots, \quad v = V_1\lambda_0 + V_2\lambda_0^2 + \cdots, \quad \cdots \tag{7}$$

where $U_1, U_2, \cdots, V_1, V_2, \cdots$ are constants.

Finally, substituting the values of $u, v, \cdots$ from Eq. (7) into the first equation of Eq. (5), ω is determined. This completes the determination of all related function for the proposed period using the relation $T = \frac{2\pi}{\omega}$.