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Current–voltage characteristics of triple-barrier Josephson junctions

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ABSTRACT

Current–voltage characteristics of triple-barrier Josephson junctions are analytically and numerically studied. In the presence of a constant current bias and for homogeneous Josephson coupling of all layers, these systems behave exactly as ordinary Josephson junctions, despite their non-canonical current-phase relation. Deviation from this behaviour is found for inhomogeneous Josephson coupling between different layers in the device. Appearance of integer and fractional Shapiro steps are predicted in the presence of r. f. frequency radiation. In particular, the amplitudes of these steps are calculated in the homogeneous case as clear footprints of the non-canonical current-phase relation in these systems.

1. Introduction

Josephson junctions (JJs) have a great variety of applications [1]. The most diffuse use of these superconducting elements can probably be recognized in the realization of quantum interference devices [2]. Usually the latter ultra-sensitive magnetic field sensors are fabricated utilizing conventional JJs. However, double or multi-barrier JJs have been also proposed as elements of Superconducting Quantum Interference Devices (SQUIDs) [3,4]. It is therefore important to study the properties of the latter types of junctions and, in particular, the current–voltage (*I–V*) characteristics of triple-barrier Josephson junctions (TBJJs).

We recall that the current–voltage (*I–V*) characteristics of a single-barrier Josephson Junction with negligible capacitive parameter can be analytically derived by means of the Resistively Shunted Junction (RSJ) model [1]. In this non-hysteretic limit, in fact, the dynamical equation of a JJ can be written as follows

$$\frac{d\phi}{d\tau} + \sin\phi = i_B, \qquad (1)$$

where, on the right hand side (r.h.s.) of (1), i_B is the bias current value I_B normalized to the maximum Josephson current I_{J0}; on the left hand side (l.h.s.), ϕ is the superconducting phase difference between the two junction electrodes and $\tau = \frac{2\pi R I_{J0}}{\Phi_0} t$ is the normalized time, R and Φ_0 being the resistive parameter of the JJ and the elementary flux quantum, respectively. The current-phase relation (CPR) of a canonical JJ can be written as $i_J = \sin\phi$, where i_J is the current flowing in the ideal Josephson element normalized to I_{J0}, so that the second addendum of the l.h.s. of (1) can be identified with this term. It is thus well known that the nonlinear ordinary differential equation (1) can be integrated by means of separation of variables and that the resulting *I–V* characteristics can be described by the following simple expression [1]:

$$i = \pm\sqrt{\langle v\rangle^2 + 1}, \qquad (2)$$

where i is the current, normalized to I_{J0}, flowing in the JJ, the plus and minus signs referring to the positive and negative voltage branches, respectively, and where $v = \frac{V}{RI_{J0}}$ is the normalized tension across the JJ, the argument in the triangular brackets being time averaged.

As for TBJJs, we first notice that their CPR is different from the usual $\sin\phi$ dependence. In fact, these four-layer systems have a CPR which can be deduced from the behaviour of double-barrier Josephson junctions (DBJJs). The latter systems have been experimentally investigated by Nevirkovets et al. [5,6]. Integer and fractional Shapiro steps were detected by the latter authors, so that deviations from the $\sin\phi$ dependence of the CPR can be hypothesized. A microscopic theory of DBJJs confirming the existence of non-sinusoidal CPRs in DBJJs has been developed by Brinkman et al. [7]. By applying a semi-classical model [8] derivable from the well known Feynman and Ohta's models for a JJ [9,10], it can be confirmed that, in the case the maximum Josephson currents, I_1 and I_2, in the two JJs of the trilayer system are different, the CPR of the DBJJ can be written as follows

$$I = I_0\left[\gamma + \frac{(1-\varepsilon^2)}{2\sqrt{1-(1-\varepsilon^2)\sin^2\frac{\phi}{2}}}\right]\sin\phi, \qquad (3)$$

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where I_0 is the average value of I_1 and I_2, and where $\varepsilon = \frac{I_2 - I_1}{2I_A}$, γ being proportional to the coupling energies between the two outermost electrodes in the DBJJ. In the limit of very small values of ε, i. e. for $I_2 \approx I_1$, the above expression reduces to the following [11]

$$I = I_0\left[\gamma \sin\phi + \mathrm{sgn}(\cos\tfrac{\phi}{2})\sin\frac{\phi}{2}\right]. \tag{4}$$

Therefore, the following expression for the CPR of a TBJJ can be argued:

$$I = I_0\frac{1-\varepsilon^2}{2}\frac{\sin\phi}{\sqrt{\varepsilon^2 + (1-\varepsilon^2)\cos^2\frac{\phi}{2}}}, \tag{5}$$

in which only the second addendum of the r.h.s. of Eq. (3) is retained. In Eq. (5) the absence of a term similar to the first addendum in the r.h.s. of Eq. (3) is due to the fact that, when considering nearest and next nearest neighbour interactions, the two superconducting layers S_2 and S_3 act as a single superconducting system interacting with S_1 and S_4. In a TBJJ the intermediate layers S_2 and S_3 do not allow, however, direct coupling between the outermost layers as it happens in a DBJJ so that the $\sin\phi$ term disappears. This argument is also supported by strict application of the Feynman's and Ohta's semiclassical model to the four-layer system [12]. Form the latter analysis we are able to extract the meaning of the parameter ε. In fact, we may define $I_1 = \frac{4e}{\hbar}(K_1\sqrt{N_1N_2} + \tilde{K}_1\sqrt{N_1N_3})$ and $I_2 = \frac{4e}{\hbar}(K_3\sqrt{N_3N_4} + \tilde{K}_2\sqrt{N_2N_4})$, where the quantities N_k are the number of superconducting electron pairs in the k-th electrode and the constant in front of $\sqrt{N_jN_k}$ represents the coupling energy between the electrodes j and k.

In the present work, by starting with the CPR in (5), we study the *I–V* characteristics of triple-barrier Josephson junctions. We first consider a homogeneous system ($\varepsilon = 0$). and analytically determine that, in this case, the *I–V* characteristics of TBJJs are given by Eq. (2) in the presence of a constant current bias. For inhomogeneous Josephson coupling ($\varepsilon \neq 0$) numerical evaluation of *I–V* characteristics are made; deviations of these curves from the analytically determined characteristics for $\varepsilon = 0$ are seen to be compatible with the expression of maximum Josephson current $I_{MAX} = (1-\varepsilon)I_0$. In the presence of a r. f. radiation integer and fractional Shapiro steps arise in the *I–V* characteristics. Expressions of the semi-amplitudes of these steps for $\varepsilon = 0$ are determined by means of a semi-analytic approach. Numerical evaluation of *I–V* curves are performed.

2. *I–V* characteristics in the presence of a constant current bias

Let us consider the CPR of TBJJ given in Eq. (5). In the particular case of $\varepsilon = 0$, we obtain:

$$I = I_0 \sin\frac{\phi}{2}\,\mathrm{sgn}\left(\cos\frac{\phi}{2}\right), \tag{6}$$

where

$$\mathrm{sgn}\left(\cos\frac{\phi}{2}\right) = \begin{cases} 1 & \text{if} \quad \cos\frac{\phi}{2} > 0 \\ -1 & \text{if} \quad \cos\frac{\varphi}{2} < 0 \end{cases}$$

Therefore, if we consider values of ϕ in the interval $[-\pi, \pi]$, we notice that $\cos\frac{\phi}{2} > 0$, so that $\mathrm{sgn}(\cos\frac{\phi}{2}) = 1$. In this way, we have

$$I = I_0 \sin\frac{\phi}{2} \tag{7}$$

for all values of ϕ in $[-\pi, \pi]$. Adopting the Resistively Shunted Josephson (RSJ) model we obtain the ordinary differential equation:

$$\frac{d\phi}{d\tau} + \sin\frac{\phi}{2} = i_B. \tag{8}$$

As in the case of a JJ, we can solve Eq. (8) by separation of variables, so that:

$$\int_{\phi_0}^{\varphi(\tau)} \frac{d\phi}{i_B - \sin\frac{\varphi}{2}} = \tau \tag{9}$$

where we take the integration interval $[\phi_0, \phi(\tau)]$ inside the interval $[-\pi, \pi]$ so that Eq. (7) holds. In order to solve the integral in the l.h.s. of Eq. (9), we can make the substitution $t = \tan\frac{\varphi}{4}$ to get:

$$\int_{\phi_0}^{\phi(\tau)} \frac{d\varphi}{i_B - \sin\frac{\varphi}{2}} = \frac{4}{\sqrt{i_B^2 - 1}}\left[\tan^{-1}\left(\frac{i_B\tan\frac{\varphi}{4} - 1}{\sqrt{i_B^2 - 1}}\right)\right]_{\phi_0}^{\phi(\tau)}. \tag{10}$$

By now setting $\tan^{-1}\left(\frac{i_B\tan\frac{\phi_0}{4} - 1}{\sqrt{i_B^2 - 1}}\right) = a_0$ and recalling that the integral in the l.h.s. of (10) is equal to τ, we have:

$$\tan^{-1}\left(\frac{i_B\tan\frac{\phi}{4} - 1}{\sqrt{i_B^2 - 1}}\right) = \frac{\sqrt{i_B^2 - 1}}{4}\tau + a_0 \tag{11}$$

By applying the tangent function to both sides of Eq. (11) we obtain:

$$\tan\frac{\phi}{4} = \frac{\sqrt{i_B^2 - 1}}{i_B}\tan\left(a_0 + \frac{\sqrt{i_B^2 - 1}}{4}\tau\right) + \frac{1}{i_B} \tag{12}$$

Knowing that the tangent is a periodic function of period π, we may write:

$$\phi(\tau) = 4\tan^{-1}\left\{\frac{1}{i_B}\left[\sqrt{i_B^2 - 1}\tan\left(\frac{\sqrt{i_B^2 - 1}}{4}\tau + a_0\right) + 1\right]\right\} + 4k\pi \tag{13}$$

where k is an integer. The result of Eq. (13) is a crucial one in obtaining the correct form of the *I–V* characteristics. In fact, while the phase slip in JJs is of 2π, in TBJJs we notice that it doubles to 4π over a pseudo-period T given by the period of the tangent function in (13):

$$T = \frac{4\pi}{\sqrt{i_B^2 - 1}}. \tag{14}$$

In this way, we can obtain an expression of the average voltage $\langle v\rangle$ as follows:

$$\langle v\rangle = \frac{1}{T}\int_0^T \frac{d\phi}{d\tau}d\tau = \frac{4\pi}{T}. \tag{15}$$

Considering now Eqs. (14) and (15), we can get Eq. (2) and thus, the *I–V* characteristics of TBJJs in the case $\varepsilon = 0$. By this analysis we can conclude that the *I–V* characteristics of homogeneous TBJJs are equal to those of a JJ.

When we consider $\varepsilon \neq 0$, we recur to numerical analysis, by first integrating the non-linear ordinary differential equation

$$\frac{d\phi}{d\tau} + \frac{1-\varepsilon^2}{2}\frac{\sin\phi}{\sqrt{\varepsilon^2 + (1-\varepsilon^2)\cos^2\frac{\phi}{2}}} = i_B. \tag{16}$$

Numerical solutions for $\varepsilon = 0.15$ and $i_B = 0.9, 1.1, 1.3, 1.5$ of $\phi(\tau)$ and $\frac{d\phi}{d\tau}$ as determined by Eq. (16) are represented in Fig. 1(a) and (b) respectively.

Successively, we evaluate, for single values of i_B, the average voltage $\langle v\rangle$, by means of the numerically determined function $\phi(\tau)$ and its derivative. In this way, in Fig. 2 we show a curve for $\varepsilon = 0.15$ along with the $\varepsilon = 0$ curve given by Eq. (1). In the latter figure we notice that the $\varepsilon = 0.15$ curve lies below the $\varepsilon = 0$ curve. This feature can be understood by generalizing the zero voltage