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ORIGINAL ARTICLE

# An analytical algorithm for nonlinear fractional Fornberg–Whitham equation arising in wave breaking based on a new iterative method

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**Abstract** This work suggests a new analytical technique called the fractional homotopy analysis transform method (FHATM) for solving fractional Fornberg–Whitham equation. The fractional homotopy analysis transform method is an innovative adjustment in Laplace transform algorithm (LTA) and makes the calculation much simpler. In this paper, it can be observed that auxiliary parameter $\hbar$, which controls the convergence of the FHATM approximate series solutions, also can be used in predicting and calculating multiple solutions. This is basic and more qualitative difference in analysis between FHATM and other methods. The solutions obtained by proposed method indicate that the approach is easy to implement and computationally very attractive. The proposed method is illustrated by solving numerical example.



## 1. Introduction

In the past few decades, fractional differential equations (FDEs) have been the focus of many studies due to their frequent appearances in various applications in fluid mechanics, viscoelasticity, biology, physics, electrical network, control theory of dynamical systems, Chemical physics, optics and signal processing, as they can be modeled by linear and non-linear fractional order differential equations. Fractional order ordinary differential equations, as generalizations of classical integer order ordinary differential equations. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Half-order derivatives and integrals proved to be more useful for the formulation of certain electrochemical problems than the classical models [1–4]. Nonlinear fractional partial differential equations have many applications in various fields of science and engineering such as fluid mechanics, thermodynamics, mass and heat transfer, and micro-electro mechanics system.

The Fornberg–Whitham equation was first proposed for studying the qualitative behavior of wave breaking [5]. The Fornberg–Whitham equation can be written as

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$$u_t - u_{xxt} + u_x = uu_{xxx} - uu_x + 3u_x u_{xx}, \quad t > 0, \tag{1.1}$$

where $u(x,t)$ is the fluid velocity, $t$ is the time, and $x$ is the spatial co-ordinate. In 1978, Fornberg and Whitham obtained a peaked solution $u(x,t) = Ae^{-\frac{1}{2}\left|x-\frac{4}{3}t\right|}$ with an arbitrary constant A [6]. The different types of Fornberg–Whitham equations in physics have been solved by Abidi and Omrani [7], Gupta and Singh [8], Lu [9], Sakar et al. [10]. Chen et al. [11], Yin et al. [12], Zhou and Tian [13], He et al. [14], Fan et al. [15], Jiang and Bi [16].

In this paper, the homotopy analysis transform method (HATM) basically illustrates how the Laplace transform can be used to approximate the solutions of the linear and nonlinear fractional differential equations by manipulating the homotopy analysis method. The proposed method is coupling of the homotopy analysis method and Laplace transform method. The main advantage of this proposed method is its capability of combining two powerful methods for obtaining rapid convergent series for fractional partial differential equations. Homotopy analysis method (HAM) was first proposed and applied by Liao [17–20] based on homotopy, a fundamental concept in topology and differential geometry. The HAM is based on construction of a homotopy which continuously deforms an initial guess approximation to the exact solution of the given problem. An auxiliary linear operator is chosen to construct the homotopy and an auxiliary parameter is used to control the region of convergence of the solution series. The HAM provides greater flexibility in choosing initial approximations and auxiliary linear operators and hence a complicated nonlinear problem can be transformed into an infinite number of simpler, linear sub problems, as shown by Liao and Tan [21]. The HAM has been successfully applied by many researchers for solving linear and non-linear partial differential equations [22–34]. In recent years, many authors have paid attention to studying the solutions of linear and nonlinear partial differential equations by using various methods with combined the Laplace transform. Among these are the Laplace decomposition methods [35,36], homotopy perturbation transform method [37–41]. Recently, many researchers [42–45] have applied to obtain the solutions of the many differential and integral equations by coupling of homotopy analysis and Laplace transform method. Recently, many researchers [46–48] have solved differential and integral equations by using optimal homotopy asymptotic method.

This paper is committed to the study of time fractional Fornberg–Whitham equation by using new fractional homotopy analysis transform method. Using the appropriate initial condition, the approximate analytical solutions for different fractional Brownian motions and also for standard motion are obtained. The variation on the approximate solutions is depicted through graphically and error analysis shows the accuracy of the approximate analytical solutions. This equation can be written in time fractional operator form as

$$D_t^\alpha u - D_{xxt}u + D_x u = uD_{xxx}u - uD_x u + 3D_x u D_{xx} u, \quad t > 0, \quad 0 < \alpha \leqslant 1, \tag{1.2}$$

with initial condition $u_0(x,t) = \frac{4}{3}\exp\left(\frac{x}{2}\right)$ and $\alpha$ is parameter describing the order of the time fractional derivative and lie in the interval $(0,1]$. We remark that the exact travelling wave solution $u(x,t) = \frac{4}{3}\exp\left(\frac{1}{2}x - \frac{2}{3}t\right)$ to the above initial value problem is given by [8].

**Definition 1.1.** The Laplace transform of the function $f(t)$ is defined by

$$F(s) = L[f(t)] = \int_0^\infty e^{-st} f(t)dt. \tag{1.3}$$

**Definition 1.2.** The Laplace transform $L[u(x,t)]$ of the Riemann–Liouville fractional integral is defined as [4]:

$$L[I_t^\alpha u(x,t)] = s^{-\alpha} L[u(x,t)]. \tag{1.4}$$

**Definition 1.3.** The Laplace transform $L[u(x,t)]$ of the Caputo fractional derivative is defined as [4]

$$L[D_t^{n\alpha} u(x,t)] = s^{n\alpha} L[u(x,t)] - \sum_{k=0}^{n-1} s^{(n\alpha-k-1)} u^{(k)}(x,0), \quad n-1 < n\alpha \leqslant n. \tag{1.5}$$

## 2. Basic idea of newly fractional homotopy analysis transform method (FHATM)

To illustrate the basic idea of the FHATM for the fractional partial differential equation, we consider the following fractional partial differential equation as:

$$D_t^{n\alpha} u(x,t) + R[x]u(x,t) + N[x]u(x,t) = g(x,t), \quad t > 0, \quad x \in R, \quad n-1 < n\alpha \leqslant n, \tag{2.1}$$

where $D_t^{n\alpha} = \frac{\partial^{n\alpha}}{\partial t^{n\alpha}}$, $R[x]$ is the linear operator in $x$, $N[x]$ is the general nonlinear operator in $x$, and $g(x,t)$ are continuous functions. For simplicity we ignore all initial and boundary conditions, which can be treated in similar way. Now the methodology consists of applying Laplace transform first on both sides of Eq. (2.1), we get

$$L[D_t^{n\alpha} u(x,t)] + L[R[x]u(x,t) + N[x]u(x,t)] = L[g(x,t)]. \tag{2.2}$$

Now, using the differentiation property of the Laplace transform, we have

$$L[u(x,t)] - \frac{1}{s^{n\alpha}} \sum_{k=0}^{n-1} s^{(n\alpha-k-1)} u^k(x,0) + \frac{1}{s^{n\alpha}} L(R[x]u(x,t) + N[x]u(x,t) - g(x,t)) = 0. \tag{2.3}$$

We define the nonlinear operator

$$\eta[\phi(r,t;q)] = L[\phi(x,t;q)] - \frac{1}{s^{n\alpha}} \sum_{k=0}^{n-1} s^{(n\alpha-k-1)} u^k(x,0) + \frac{1}{s^{n\alpha}} L(R[x]u(x,t) + N[x]u(x,t) - g(x,t)), \tag{2.4}$$

where $q \in [0,1]$ be an embedding parameter and $\phi(x,t;q)$ is the real function of $x$, $t$ and $q$. By means of generalizing the traditional homotopy methods, the great mathematician Liao [17–21] construct the zero order deformation equation