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# Asymptotic expansion homogenisation and topology optimisation of cellular materials

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**Abstract**

Topology optimisation defines a set of tools associated to the modelling of an effective material domain within structural optimisation. Based on this type of optimisation, it is possible to obtain an optimal material distribution for several applications and requirements. Cellular materials are part of the most prominent materials today, both in terms of applications, and in terms of research and development. However, their potentially complex and heterogeneous structures carry some complexities, associated to the prediction of effective constitutive properties and to its design. Homogenisation procedures can provide answers for both cases. On the one hand, the asymptotic expansion homogenisation can be used to determine thermomechanical effective properties for these materials through the detailed modelling of representative unit-cells, in a flexible and accurate fashion, regardless of the type of constituent distribution. On the other hand, this homogenisation technique integrates a localisation procedure, able to obtain detailed information on the behaviour of the material within the unit-cell, giving way to local sensitivities that can be used to control optimisation procedures. This leads to a material topology optimisation approach, perfectly suited for the design of this type of material. Within this scope, this work focuses on the analysis of effective thermomechanical material properties of cellular materials designed with topology optimisation procedures.


*Keywords:* representative unit-cell; homogenisation; effective properties; optimisation; cellular materials.

## 1. Introduction

The methods presented in this work are primarily developed for complex materials with periodical material distributions. Cellular materials, namely metallic foams, are commonly part of this group, with an average constituent distribution that closely approximates a periodic lattice. This creates an opportunity to use powerful methods such as the Asymptotic Expansion Homogenisation (AEH) and closely related optimisation methods, making the numerical study and development of these materials free from restrictive micromechanics models which depend on specific types of constituent distributions. This provides the capability of accurately predicting properties for any phase distribution within a cellular material. Moreover, AEH allows for the use of an inverse method, called localisation, which brings macroscale homogenised results back to the detailed heterogeneous material microstructure. This is, in fact, the step that gives in turn way to the optimisation approach called inverse homogenisation. This paper describes the basis of the application of these methods to cellular materials. It focuses on the topology

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optimisation of microstructures for given typical mechanical and thermal requirements. Some numerical strategies are also discussed, as well as a close look at the effective properties of the achieved microstructures. The authors use an in-house developed code to solve all the problems presented along this work [1].

## 2. Material processing framework

### *2.1. Asymptotic expansion homogenisation*

The asymptotic expansion homogenisation provides an efficient tool to determine global thermomechanical properties based on a Representative Unit-Cell (RUC) of the local material distribution. Moreover, it provides localisation tools and sensitivity information useful for optimisation methodologies. Mathematical formulation and implementation details can be studied in detail in several references [1-5]. Within the scope of this document, the most important part of the AEH methodology is related to the homogenised constitutive matrices of the materials.

Within the topology optimisation approach of this work, the constituent microscale properties are controlled by the base material, with an elasticity constitutive matrix $D_0$, and the Solid Isotropic Material with Penalisation (SIMP) method [6, 7]. In this, the local material densities, $\mu$, vary between 0 (void) and 1 (base material). Moreover, the densities are penalised, using a penalty value $p$. Thus, the constitutive matrix $D_k$ for an element $k$ of a material microscale domain Y can be defined as

$$\mathbf{D}_k = \mathbf{D}(\mu_k) = \mu_k^p \mathbf{D}_0 \,. \tag{1}$$

In this sense, the homogenised matrix can be written, based on a finite element discretisation and using a quadratic form (variational) [8], as

$$\mathbf{D}^{\mathrm{h}} = \sum_{k=1}^{n_e} \frac{Y^k}{Y} (\mathbf{I} - \mathbf{B}^k \boldsymbol{\chi}^k)^{\mathrm{T}} \mu_k^p \mathbf{D}^0 (\mathbf{I} - \mathbf{B}^k \boldsymbol{\chi}^k) \,, \tag{2}$$

where $Y^k$, Y and $n_e$ are the volume of the generic finite element $k$, the total volume for the RUC and the total number of finite elements, respectively. I is the identity matrix and B is the matrix of the derivatives of the finite element shape functions. $\chi$ is the matrix of displacement correctors, which contains the eigendeformations of the representative periodic geometry. These changes to local properties, with the influence of the density interpolation, must also be used for the equations that define the local homogenisation problems [4].

Following the same approach, the homogenised thermal conductivity matrix $k^h$ can be defined as

$$\mathbf{k}^{\mathrm{h}} = \sum_{k=1}^{n_e} \frac{Y^k}{Y} \left(\mathbf{I} - \tilde{\mathbf{M}}^k \boldsymbol{\Upsilon}^k\right)^{\mathrm{T}} \mu_k^p \mathbf{k}^0 \left(\mathbf{I} - \tilde{\mathbf{M}}^k \boldsymbol{\Upsilon}^k\right) . \tag{3}$$

$k_0$ is the matrix of thermal conductivity coefficients for the base material and M the matrix of shape elements. $\Upsilon$ is the matrix of thermal conductivity correctors and contains the temperature eigendeformations of the representative periodic geometry. Uncoupled thermoelastic behaviour may also be studied with the AEH, defining the homogenised vector of thermal expansion by

$$\boldsymbol{\beta}^{\mathrm{h}} = \sum_{k=1}^{n_e} \frac{Y^k}{Y} \left(\mu_k^{2p} \boldsymbol{\beta}^0 - \mu_k^p \mathbf{D}^0 \mathbf{B}^k \boldsymbol{\Psi}^k\right) . \tag{4}$$

This is related to the homogenised vector of thermal expansion coefficients, obtained as

$$\boldsymbol{\alpha}^{\mathrm{h}} = \left(\mathbf{D}^{\mathrm{h}}\right)^{-1} \boldsymbol{\beta}^{\mathrm{h}} . \tag{5}$$

The material properties for each finite element are obtained from the base material properties,

$$\boldsymbol{\beta}_0 = \mathbf{D}_0 \boldsymbol{\alpha}_0 \tag{6}$$

according to its density value, as

$$\boldsymbol{\beta}_k = \mu_k^p \mathbf{D}_0 \mu_k^p \boldsymbol{\alpha}_0 * \mu_k^{2p} \boldsymbol{\beta}_0 \tag{7}$$

$\boldsymbol{\Upsilon}$ is the matrix of thermal expansion correctors and contains the expansion eigendeformations of the representative periodic geometry. Note that if the corrector gradients are zero, the homogenised quantities become the volume average of the properties of the microscale constituents.

### *2.2. Local optimisation*

In structural mechanics, there are alternative methods to determine the optimal material distribution for a given application. This is also valid when approaching specifically the material distribution within a representative volume. One possibility is to approach the microstructural problem with the topology optimisation method commonly used for macrostructural problems. In this case, the objective function can be the work of external loads, defining a compliance measure to be minimised. For the