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# An analytical method for the vibration and buckling of functionally graded beams under mechanical and thermal loads

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ABSTRACT

An analytical method for vibration and buckling behaviours of Functionally Graded (FG) beams with various boundary conditions under mechanical and thermal loads is presented. Based on linear strain-displacement relations, equations of motion and essential boundary conditions are derived from Hamilton's principle. In order to account for thermal effects, three cases of the temperature rise through the thickness, which are uniform, linear and nonlinear, are considered. The exact solutions are derived using the state space approach. Numerical results are presented to investigate the effects of boundary conditions, temperature distributions, material parameters and slenderness ratios on the critical temperatures, critical buckling loads, and natural frequencies as well as load-frequencies curves, temperature-frequencies curves of FG beams under thermal/mechanical loads. The accuracy and effectiveness of proposed model are verified by comparison with previous research.



## 1. Introduction

Functionally graded materials (FGMs) are a class of composites materials in which the material properties gradually vary in a specific direction. By this way, the distribution of strength and stiffness can be customised in a designable manner and the delamination which may occur in laminated composites can be avoided. Due to the excellent properties in mechanical and thermal behaviours, a wide range of application for functionally graded (FG) structures can be found in different fields, leading to the intensive study in many types of FG structures in the last three decades.

By using various theories such as The Classical Beam Theory (CBT), First-order Beam Theory (FOBT), Higher-order Beam Theory (HOBT), Quasi-3D beam theory and Carrera Unified Formulation (CUF), many numerical methods have been developed to deal with vibration and buckling behaviours of FG beams under mechanical/thermal loads. Some of popular numerical approaches are Lagrange multipliers, Rayleigh Ritz method, dynamic stiffness formulation, Chebyshev collocation method, finite element method and differential quadrature method [1–15]. For analytical approaches, a Navier solution has been widely used to study various mechanical behaviours of simply supported beams [16–20]. In addition, another analytical solution based on the state space approach, which can deal with different boundary conditions, was proposed by Khdeir and Reddy [21–23] to study the behaviour of cross-ply laminated beams. This approach was also applied for the vibration analysis of FG and FG sandwich beams [24,25]. Regarding the thermal environment, FG beams can be designed in a smart way to adapt the environment changes, which results in a good attention in studying such behaviours. Sankar and Tzeng [26] used CBT to study the thermal stresses of simply supported FG beams. The FOBT was employed to investigate various behaviours of FG beams such as dynamic responses under a moving load [27], thermal stability with non-linear hardening elastic foundations [28], thermal dynamic buckling [29], and thermal buckling and post-buckling with non-linear elastic foundation [30]. Wattanasakulpong et al. [31] used Ritz method based on the HOBT to study the buckling and vibration of FG beams, however, it was limit on uniform temperature distribution only. Based on CUF, Giunta et al. [32] developed Navier solution to analyse the static behaviour of FG beams under

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thermo-mechanical loads. However, as far as the authors are aware, there is no analytical solution for vibration and buckling of FG beams using HOBT with various boundary conditions under mechanical/thermal loads in a unitary manner. In addition, effects of various temperature distributions on natural frequencies and critical temperatures of FG beams are also need further studies. As a result, it is also the main objective of this paper. Based on linear strain-displacement relations, equations of motion and the essential boundary conditions are derived from Hamilton's principle. State space-based analytical approach is used to obtain closed-form solutions for FG beams with various configurations. Three cases of the temperature rise through the thickness, which are uniform, linear and nonlinear, are considered. Numerical results are presented for FG beams with various boundary conditions, temperature distributions and slenderness ratios to investigate the critical temperatures/loads, and natural frequencies as well as load-frequencies curves, temperature-frequencies curves. The accuracy and effectiveness of proposed model are verified by comparison with previous research.

## 2. Theoretical formulation

### 2.1. Functionally graded beams and temperature-dependent material properties

Consider a FG beam made from metal and ceramic with the span of $a$ and rectangular cross-section of $b \times h$, as shown in Fig.1. Volume fraction of ceramic is given by power law distribution:

$$V_c(z) = \left(\frac{z}{h} + \frac{1}{2}\right)^p \tag{1}$$

where $p$ is the material parameter.

The thermo-elastic material properties are considered as a function of temperature $T$ and can be calculated for ceramic and metal as described in Ref. [33]:

$$P(T) = P_0\left(P_{-1}T^{-1} + 1 + P_1T + P_2T^2 + P_3T^3\right) \tag{2}$$

where $P$ denotes Young's modulus $E$, mass density $\rho$ and thermal expansion coefficient $\alpha$, respectively. $P_{-1}$,$P_1$,$P_2$ and $P_3$ are the temperature dependent coefficients, which are listed in Table 1 for various materials. Fig. 2 presents the material properties of ceramics and metals with respect to the temperature change. Based on the power rule together with the temperature-dependence described in Eq. (2), the typical material properties $P(z,T)$ of beam through the thickness are described as:

$$P(z,T) = [P_c(T) - P_m(T)]V_c(z) + P_m(T) \tag{3}$$

The material properties are calculated by Eq. (2) for ceramic and metal at the specific temperature and followed by Eq. (3) to obtain the values at $z$. It should be noticed that the Poisson's ratio$\upsilon$ is evaluated as the average of ceramic and metal values at $T_0 = 300\ K$.

### 2.2. Temperature distribution

#### 2.2.1. Uniform Temperature Rise (UTR)

The temperature of the whole beam is assumed uniform and increased from $T_0 = 300K$ to the current value. It means that the temperature at a point is $T(z)$=$T_0 + \Delta T$, where $\Delta T$ is the temperature rise.

#### 2.2.2. Linear Temperature Rise (LNR)

The temperature in the ceramic and metal faces of FG beam is assumed to be $T_c$ and $T_m$. In this case, the temperature on the metal surface is supposed to be $T_m = 305K$, whereas on the ceramic surface it is surged to $T_c = T_0 + \Delta T$. With the assumption of linear distribution, the temperature through the thickness can be determined as:

$$T(z) = T_m + \Delta T\left(\frac{1}{2} + \frac{z}{h}\right) \tag{4}$$

#### 2.2.3. Non-linear Temperature Rise (NLNR)

The applied temperature is similar to the case of LNR; however, the temperature distribution is set to follow the heat conduction rule and obtained by solving the steady state equation [27] as:

$$T(z) = T_m + \frac{T_c - T_m}{C} \times \left( V_f - \frac{K_{cm}}{(p+1)K_{cm}}V_f^{p+1} + \frac{K_{cm}^2}{(2p+1)K_m^2}V_f^{2p+1} - \frac{K_{cm}^3}{(3p+1)K_m^3}V_f^{3p+1} + \frac{K_{cm}^4}{(4p+1)K_m^4}V_f^{4p+1} - \frac{K_{cm}^5}{(5p+1)K_m^5}V_f^{5p+1} \right) \tag{5}$$

where $C = 1 - \frac{K_{cm}}{(p+1)K_{cm}} + \frac{K_{cm}^2}{(2p+1)K_m^2} - \frac{K_{cm}^3}{(3p+1)K_m^3} + \frac{K_{cm}^4}{(4p+1)K_m^4} - \frac{K_{cm}^5}{(5p+1)K_m^5}$; $K_{cm} = K_c - K_m$and $K_c$ and $K_m$ are the thermal conductivity of ceramic and metal calculated at the surfaces.

### 2.3. Kinematics

Assuming that the deformation of FG beam is only in the $x - z$ plane and let $u(x,z,t)$ and $w(x,z,t)$ be the axial and transverse displacements at an arbitrary point. These components can be expressed in terms of the displacement components on the neutral line as:

$$\begin{aligned} u(x,z,t) &= U(x,t) - zW'(x,t) + \left(z - \frac{4z^3}{3h^2}\right)\phi_x(x,t) \\ &= U(x,t) - zW'(x,t) + f(z)\phi_x(x,t) \end{aligned} \tag{6a}$$

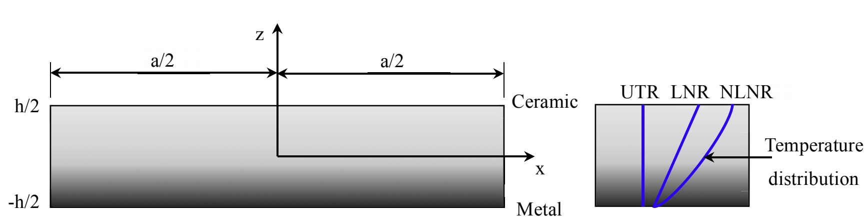


**Fig. 1.** Coordinates of FG beam and temperature distributions.