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Stress intensity factors and energy release rate for anisotropic plates based on the classical plate theory

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ABSTRACT

The stress and displacement fields near the tip of a through crack in an anisotropic elastic plate are examined according to the classical plate theory. The plate is assumed to be subjected to bending moments, twisting moments, or transverse shear forces. In particular, the stresses at the prolongation of the crack and the relative crack face displacements are derived. The results are used to define stress intensity factors which are consistent with those for isotropic material in Ref. [6]. An explicit expression for the energy release rate in terms of the stress intensity factors is obtained using the path-independent *J*-integral. Analytic solutions are given for the stress intensity factors of a crack in an infinite plate under uniform bending moments, twisting moments, or transverse shear forces. It is shown that the stress intensity factors are zero and the stresses at the crack tip are finite if only twisting moments are applied. A universal relationship between the classical theory stress intensity factors and the Reissner theory stress intensity factors for thin orthotropic plates under symmetric bending is also derived.

1. Introduction

The stress intensity factors introduced by Irwin [1] play an important role in linear elastic fracture mechanics. They represent the intensity of the local singular stress field near a crack tip and are related to the global energy release rate per unit of newly created fracture surface area. The fracture parameters may be used to evaluate the strength, tolerant flaw size or fatigue life of cracked bodies. There are numerous research works on the stress intensity factors for plane extension problems [2,3,4]. Similar works for bending problems are considerably less.

The stress and displacement fields near the tip of a through crack in a thin elastic plate under bending were first obtained by Williams [5] based on the classical plate theory. Sih, Paris and Erdogan [6] introduced two bending stress intensity factors to characterize the crack-tip fields. Analytic solutions of the stress intensity factors were derived for an infinite plate containing a finite crack subjected to uniform bending, twisting and shearing. Those due to twisting and shearing were later corrected by Zender and Hui [7].

The crack-tip stress field according to the classical plate theory is not entirely the same as that derived from the elasticity theory. The in-plane stresses in the classical plate theory exhibit the same $r^{-1/2}$ singularity, where r is the distance from the crack tip, as that predicted by the elasticity theory. However, the transverse shear stresses have a stronger $r^{-3/2}$ singularity. Moreover the angular distributions of stresses depend on Poisson's ratio. The discrepancy occurs because of the inability of the classical plate theory to satisfy the traction-free conditions for in-plane and transverse shear stresses independently at the crack faces. To remedy the inadequacy, the Reissner plate theory, which allows for transverse shear deformations, was used to obtain the crack-tip stress field which is consistent with that predicted by the elasticity theory [8]. Although the classical theory and the Reissner theory yield different crack tip fields, Simmonds and Duva [9] showed that the energy release rates calculated from the two theories are the same for thin plates. Invoking Simmonds and Duva's result, Hui and Zender [10] obtained a universal relationship between the classical theory stress intensity factors and the Reissner theory stress intensity factors for thin plates. Hui and Zender also showed that when correlating fracture toughness data for thin elastic plates it is sufficient and, in many cases, preferable to use the classical theory rather than the Reissner theory. This is because even a small amount of crack tip plasticity will render the Reissner theory fields invalid due to its small region of dominance relative to the plate thickness. The dominance of the classical plate theory is larger.

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The aforementioned works are for isotropic plates and cannot be applied to treat composite plates, which are anisotropic. Similar works on anisotropic plates, however, appear to be limited. For the classical plate theory, Sih and Chen [11] provided the crack-tip stress field for anisotropic plates, containing parameters with rather complicated expressions. Neither the definition of stress intensity factors nor the energy release rate was given. Explicit closed form solutions for an anisotropic plate containing an elliptical hole subjected to out-of-plane bending moments were obtained by Hsieh and Hwu [12]. Similar results for a laminated composite plate subjected to remote uniform membrane stress resultants and bending moments were given by Cheng and Reddy [13]. In Refs. [12] and [13], however, the requirement that deflection be a single-valued function was not taken into account. The solutions in Ref. [12] were modified to satisfy the single-valuedness requirement by Wu and Hsiao [14]. The solutions of the crack problems were obtained from those of the elliptic hole problems by letting the minor axis of the elliptic hole tend to zero. Chattopadhyay [15] considered the problem of an infinite orthotropic plate with a finite crack under bending or twisting moments.

For the Reissner plate theory, Wu and Erdogan [16] considered collinear cracks in an orthotropic plate under symmetric bending. Yuan and Yang [17] derived the crack-tip fields in an anisotropic plate. The crack-tip fields are analogous to those for plane stress and anti-plane deformations. An expression for energy release rate for orthotropic plates was also derived. A displacement discontinuity formulation was presented by Wen and Aliabadi [18] for modeling cracks in orthotropic plates.

It is the objective of this work to examine the crack-tip fields and extend the definition of stress intensity factors introduced in Ref. [6] to general anisotropic plates based on the classical plate theory. The plan of the paper is as follows. First a Stroh-like formalism [12] is outlined. The formulation is then used to derive the crack-tip fields and the energy release rate in terms of the stress intensity factors. The formulation is also employed to obtain several analytic solutions of the stress intensity factors for a finite crack in an infinite plate. A universal relationship between the classical theory stress intensity factors and the Reissner theory stress intensity factors for thin orthotropic plates under symmetric bending is also derived. Finally some concluding remarks are made.

In the following discussions the Latin indices range from 1 to 3, the Greek indices range from 1 to 2, summation over repeated indices is implied, and a comma in the subscript stands for partial differentiation.

2. Stroh-like formalism for classical plate theory

In the classical plate theory, the displacements are assumed as

$$u_\alpha = x_3\phi_\alpha(x_1,x_2),\; u_3 = w(x_1,x_2), \tag{1}$$

where u_α are the in-plane displacements, w is the deflection, and ϕ_α are the rotations related to w by

$$\phi_\alpha = -w_{,\alpha}. \tag{2}$$

In the absence of transverse loads on the plate surfaces the equilibrium equations are

$$M_{\alpha\beta,\alpha} = Q_\beta,\; Q_{\beta,\beta} = 0, \tag{3}$$

where $M_{\alpha\beta}$ are the moments, Q_β are the transverses shear forces. From Eq. (3), the moments and shear forces can be expressed in terms of two stress functions, ψ_1 and ψ_2, as

$$M_{11} = -\psi_{1,2},\; M_{22} = \psi_{2,1},\; M_{12} = \frac{1}{2}\left(\psi_{1,1} - \psi_{2,2}\right), \tag{4}$$

$$Q_1 = -\eta_{,2},\; Q_2 = \eta_{,1},\; \eta = \frac{1}{2}\left(\psi_{1,1} + \psi_{2,2}\right). \tag{5}$$

It is assumed that $\sigma_{33} = 0$ and that the in-plane stresses $\sigma_{\alpha\beta}$ and transverse shear stresses $\sigma_{3\alpha}$ are given by

$$\sigma_{\alpha\beta} = \frac{12x_3}{h^3}M_{\alpha\beta}, \tag{6}$$

$$\sigma_{3\alpha} = \frac{3}{2h}\left(1 - \left(\frac{2x_3}{h}\right)^2\right)Q_\alpha, \tag{7}$$

where h is the plate thickness. The constitutive equations are

$$\begin{bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \phi_{1,1} \\ \phi_{2,2} \\ 2\phi_{1,2} \end{bmatrix}, \tag{8}$$

where $D_{ik}, i,k = 1,2,6$, are the bending stiffness constants. For orthotropic materials with the symmetry planes coinciding with the coordinate planes

$$\begin{aligned} &D_{11} = \frac{h^3}{12}\frac{E_1}{1-\nu_1\nu_2},\; D_{22} = \frac{h^3}{12}\frac{E_2}{1-\nu_1\nu_2}, \\ &D_{12} = \frac{h^3}{12}\frac{\nu_1 E_2}{1-\nu_1\nu_2} = \frac{h^3}{12}\frac{\nu_2 E_1}{1-\nu_1\nu_2}, \\ &D_{66} = \frac{h^3}{12}G_{12},\; D_{16} = D_{26} = 0, \end{aligned} \tag{9}$$

where E_α, ν_α and G_{12} are, respectively, Young's moduli, Poisson's ratios, the shear modulus.

Substitution of Eqs. (4) and (5) into Eq. (8) and rearranging the terms yields [12].

$$\begin{bmatrix} \phi_{,2} \\ \psi_{,2} \end{bmatrix} = \mathbf{N}\begin{bmatrix} \phi_{,1} \\ \psi_{,1} \end{bmatrix}, \tag{10}$$

where

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix},\; \psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix},\; \mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{bmatrix}, \tag{11}$$

$$\mathbf{N}_1 = \begin{bmatrix} 0 & 1 \\ -\frac{D_{12}}{D_{22}} & -\frac{2D_{26}}{D_{22}} \end{bmatrix}, \mathbf{N}_2 = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{D_{22}} \end{bmatrix}, \tag{12}$$

$$\mathbf{N}_3 = -\begin{bmatrix} D_{11} - \frac{D_{12}^2}{D_{22}} & 2\left(D_{16} - \frac{D_{12}D_{26}}{D_{22}}\right) \\ 2\left(D_{16} - \frac{D_{12}D_{26}}{D_{22}}\right) & 4\left(D_{66} - \frac{D_{26}^2}{D_{22}}\right) \end{bmatrix}, \tag{13}$$

and $\mathbf{N}_1^T$ is the transpose of. $\mathbf{N}_1$.

Let ξ_α be the eigenvector corresponding to the eigenvalue p_a with positive imaginary part of $\mathbf{N}$, i.e.

$$\mathbf{N}\xi_\alpha = p_\alpha\xi_\alpha \;(\text{no sum on } \alpha), \tag{14}$$