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On the foundations of anisotropic interior beam theories

Anssi T. Karttunen*, Raimo von Hertzen

Department of Applied Mechanics, School of Engineering, Aalto University, FI-00076 Aalto, Finland

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ABSTRACT

This study has two main objectives. First, we use the Airy stress function to derive an exact general interior solution for an anisotropic two-dimensional (2D) plane beam. Second, we cast the solution into the conventional form of 1D beam theories to clarify some basic concepts related to anisotropic interior beams. The derived general solution provides the exact third-order interior kinematic description for the plane beam and includes the Levinson/Reddy–kinematics as a special case. By applying the Clapeyron's theorem, we show that the stresses acting as surface tractions on the lateral end surfaces of the interior beam need to be taken into account in all energy-based considerations related to the interior beam in order to avoid artificial end effects. Exact 1D interior beam equations are formed from the general 2D solution. Finally, we develop an exact interior beam finite element based on the general solution. With full anisotropic coupling, the stiffness matrix of the element becomes initially asymmetric due to the interior nature of the plane beam. By redefining the generalized nodal axial forces of the element, the stiffness matrix takes a symmetric form.

1. Introduction

Efficient use of anisotropic composite materials in mechanical design requires thorough understanding of anisotropic elasticity and accurate tools of analysis. Motivated by this, we study here a two-dimensional (2D) plane beam with anisotropic coupling effects within the framework of 2D linear elasticity.

There are two well-known complex variable formulations for 2D linearly elastic anisotropic plane problems, the Lekhnitskii and Stroh formalisms [1–3]. When it comes to 2D interior plane beam problems, where the end effects are neglected by virtue of the Saint Venant's principle, a number of more straightforward polynomial-based stress function approaches can be found in the literature, e.g. Refs. [4–6]. It is common for these classical polynomial approaches that a solution is generated only for one problem at a time. In this study, we first provide a more versatile method for 2D beams which is based on a general interior solution derived using the Airy stress function. Our approach is different from the generalization of Silverman's method [4] by Ding et al. [7] in the way that, rather than constructing stress functions for each problem separately, the focus is on solving the cross-sectional force and moment resultants along a beam for each case in the same way as in conventional beam theories, while the core of the used stress function is always the same.

There is a number of ingenious theories meant specifically for anisotropic beams, e.g. Refs. [8–13]. For a historical survey on the topic, see the book by Hodges [14]. The mentioned anisotropic beam theories are typically applicable to a wider variety of practical problems than the linearly elastic beam with a rectangular cross-section studied in this paper. However, it is commonplace in the referenced treatments to employ engineering assumptions and/or to rely on a quite heavy theoretical machinery. Therefore, it is often difficult to see the underlying structure of the developments in a fully explicit (and assumption-free) form in order to state something fundamental on anisotropic beams in general. Thus, for the remainder of this study, we view the derived 2D solution for the anisotropic plane beam as a conventional one-dimensional (1D) beam theory in order to clarify certain concepts related to anisotropic interior beams.

As the starting point for the 1D considerations, the general solution provides the exact third-order interior kinematic description for the plane beam. By "third-order kinematics" we mean that the displacement components defined at the central axis of the beam are expanded by third-order polynomials throughout the height of the beam. Many approximate beam and plate theories, which are used also in association with anisotropic composite materials, are based on similar, but assumed, displacement fields. For reviews on assumed third-order kinematics, see the works by Jemielita [15] and Reddy [16,17]. The exact interior kinematic description derived from the general solution in this paper includes, as a special case, the displacement

* Corresponding author. Tel.: +358 (0) 405272713.
E-mail addresses: anssi.karttunen@aalto.fi (A.T. Karttunen), raimo.von.hertzen@aalto.fi (R. von Hertzen).

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field which is used to derive the widely known Levinson and Reddy–Bickford beam theories [18–20].

The long-standing belief in the literature is that the Levinson beam theory is "variationally inconsistent", that is to say, it cannot be derived using the principle of virtual displacements. In contrast to this, we showed in a very recent study that the Levinson theory is actually variationally consistent [21]. The variational formulation was carried out by taking into account the fact that the stresses of the beam act as surface tractions on the lateral end surfaces of the Levinson beam. On the other hand, it was shown that the boundary layer behavior of the Reddy–Bickford beam is artificial. In the present study, the methodology utilized for the isotropic Levinson theory in Ref. [21] is further elucidated within the 2D interior framework of anisotropic elasticity.

The rest of the paper is organized in the following way. In Section 2, we formulate the anisotropic plane beam problem to which the solution is then given in terms of a stress function. The strains are calculated from the stresses under plane stress conditions and the exact 2D interior displacement field is obtained by integrating the strains. Calculation examples are presented. In Section 3, three kinematic variables defined at the central axis of the plane beam are formed from the 2D interior displacement field. Using these variables, the third-order kinematic description for the beam is given. Clapeyron's theorem is employed to facilitate energy-based considerations. Variational and vectorial approaches for interior theories are discussed and finally 1D beam equations for the 2D plane beam are obtained by a direct method. In Section 4, an exact interior beam finite element is developed both by a force-based method and from the total potential energy of the anisotropic plane beam. Conclusions are presented in Section 5.

2. General interior approach for a plane beam

2.1. Problem formulation

A 2D linearly elastic homogeneous anisotropic plane beam under a uniform pressure p is shown in Fig. 1. We have chosen the uniform load as a representative load for our developments. The beam has a rectangular cross-section of constant thickness t and the length and height of the beam are L and h, respectively. The load resultants N, M and Q stand for the axial force, bending moment and shear force, respectively, and act at an arbitrary cross-section of the beam. These cross-sectional load resultants are calculated from

$$N(x)=t\int_{-h/2}^{h/2}\sigma_x dy\,,\ M(x)=t\int_{-h/2}^{h/2}\sigma_x y dy\,,\ Q(x)=t\int_{-h/2}^{h/2}\tau_{xy} dy\,, \tag{1}$$

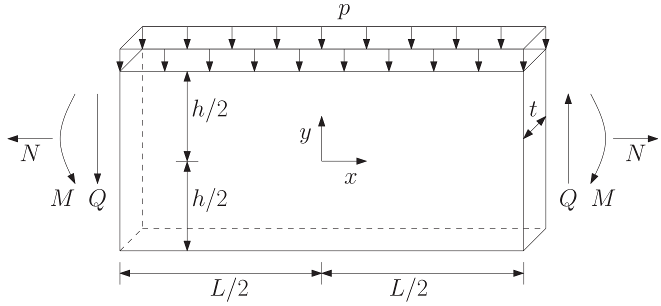

Fig. 1. 2D anisotropic plane beam under a constant uniform pressure. The load resultants act at an arbitrary cross-section of the beam. The positive directions of the load resultants are defined for later use in calculation examples and finite element developments.

which can be used to impose force and moment boundary conditions at $x = \pm L/2$. The boundary conditions on the upper and lower surfaces of the plane beam are

$$\sigma_y(x, h/2) = -p\,, \quad \sigma_y(x, -h/2) = 0\,, \quad \tau_{xy}(x, \pm h/2) = 0. \tag{2}$$

The boundary conditions are satisfied in a *strong* (pointwise) sense on the upper and lower surfaces, but at the beam ends the tractions are specified only through the load resultants and, thus, the boundary conditions are imposed only in a *weak* sense [22]. The replacement of the true boundary conditions at the beam ends by the statically equivalent weak boundary conditions (load resultants) means that the exponentially decaying end effects of the anisotropic plane beam are neglected by virtue of the Saint Venant's principle and only the interior solution of the beam is under consideration. The interior solution represents actually a beam section with fully-developed interior stresses which has been cut off from a complete beam far enough from the real lateral boundaries at which the true boundary conditions could be set. Using the Airy stress function $\Psi(x,y)$, the stresses of the plane beam are obtained from the equations

$$\sigma_x = \frac{\partial^2\Psi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2\Psi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2\Psi}{\partial x\partial y}, \tag{3}$$

which satisfy the two-dimensional equilibrium equations. To ensure compatibility, it is required that the stress function satisfies the governing equation [23].

$$s_{22}\frac{\partial^4\Psi}{\partial x^4} - 2s_{26}\frac{\partial^4\Psi}{\partial x^3\partial y} + 2(s_{12}+s_{66})\frac{\partial^4\Psi}{\partial x^2\partial y^2} - 2s_{16}\frac{\partial^4\Psi}{\partial x\partial y^3} + s_{11}\frac{\partial^4\Psi}{\partial y^4} = 0, \tag{4}$$

where s_{ij} are the elastic compliances, see Eqs. (15)–(17). The solution to the interior plane beam problem is obtained by finding a solution of Eq. (4) that satisfies the stress boundary conditions (2) of the beam.

2.2. Interior stresses

By starting from the general polynomial of the fifth degree and adapting a solution procedure outlined by Barber [22, Chap. 5] to solve the polynomial coefficients, the stress function satisfying the stress boundary conditions (2) can be found as

$$\Psi(x,y) = c_1y^2 + c_2y^3 - c_3\left(\frac{3}{4}h^2xy - xy^3 - \frac{1}{2}\frac{s_{16}}{s_{11}}y^4\right) + \Psi_q, \tag{5}$$

where c_1, c_2 and c_3 are constant coefficients and the part that depends on the nature of the applied load is

$$\Psi_q = \frac{q}{240I}\left[8\frac{s_{16}^2}{s_{11}^2}y^5 - 5x^2(h-y)(h+2y)^2\right] - \frac{q}{120Is_{11}}\left[2s_{12}y^5 + 5s_{16}\left(h^2-2y^2\right)xy^2 + s_{66}y^5\right], \tag{6}$$

where $q = pt$ is the uniform line load and $I = th^3/12$ is the second moment of the cross-sectional area. The stresses are calculated from Eq. (3), after which the load resultants are obtained from Eq. (1) and can be written as

$$N(x) = 2Ac_1 + 6Ic_3\frac{s_{16}}{s_{11}}, \tag{7}$$