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Nonlinear resonance and envelope instability of intense beam in axial symmetric periodic channel

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ABSTRACT

When an intense charged particle beam propagates through a given periodic focusing channel, it will experience the phenomena of nonlinear resonance, collective instability or chaotic motion with different conditions. In this paper, the collective envelope instability mechanisms are studied for symmetric beam propagation in an axially symmetric periodic channel. The beam is characterized as collectively stable if there exists a stable fixed point (SFP) located at the matched beam condition $(r_m, 0)$ in (r, p_r) phase space. It is found that the well-known collective envelope instability is dynamically related to the period-two orbits bifurcation of the matched SFP, meanwhile the unique stable SFP turns into an unstable saddle-node, surrounded by 1/2 resonance islands. However, higher orders of resonance $(l/n,\ n>2)$ coming from period-*n* bifurcation will not lead to collective beam instability because a new SFP emerges immediately upon the bifurcation process. The orders of SFP bifurcation is numerically depicted by the envelope tune $\nu=\phi/360$, where ϕ is the eigenphase of the *Poincaré* tangent map $T(s)$ in one focusing period at SFP, as functions of depressed phase advance. With strong space charge, due to these resonances from SFP bifurcation could be overlapped, mismatched beam would even show chaotic motion. For specific parameters, regular orbits, resonance islands, chaotic regions formed by resonance overlapping are clearly depicted with frequency analysis and Lyapunov spectral exponents, a method that may prove useful when extended to higher phase-space dimensions.

1. Introduction

The role of space charge has generated adequate attention in recent years due to the increasing interest in high power linear as well as circular accelerators [1–3]. Clear understanding of the basic physics for nonlinear space charge dominated problems is of great importance both in the area of theoretical and experimental study for accelerator researchers. The analytical study of the related problems began in the late 1950s when the self-consistent Kapchinskij–Vladimirskij (KV) [4] distribution was established. In the 1970s, Sacherer demonstrated the general form of the rms envelope equation [5], and proved that the linear part of the self-field depends mainly on the rms size of the distribution and only weakly on its exact form. This encouraged researchers in this area to explore deeply on topics such as beam collective instability [6–8], nonlinear dynamics of beam envelope [9–16] and beam halo formation [17–19] in the following years.

Basically, in the subject of high intensity beam collective instability in periodic channels, two methods are normally used for modeling the beam instability: the Vlasov–Poisson description [7] and envelope Hamiltonian description [9,10,20]. In the Vlasov–Poisson description, the beam collective modes obtained with the perturbed space charge potential suggests that designed zero beam current phase σ_0 should be less than 90° to avoid the 2nd order mode, less than 60° to avoid the 3rd order mode, less than 45° to avoid the 4th order mode, etc. In the rms-matched envelope Hamiltonian description, the perturbed 2.5D rms envelope equations only show the 90° unstable stop band, named as collective envelope instability, which is the same as 2nd even mode in Vlasov–Poisson description. Whereas, in the envelope Hamiltonian, if beam was not rms matched, other phenomena such as envelope nonlinear resonances, unpredictable chaos behavior would take place [11–16]. In this paper, we focus on the envelope Hamiltonian description.

To depict the nonlinear characteristic and the collective envelope instability clearly, we perform a steady-state analysis of the initial symmetric beam $(\tilde{\epsilon}_x=\tilde{\epsilon}_y)$ envelope evolution in an axially symmetric periodic channels, which is simply described as a 1.5D nonautonomous system. Compared with the 2.5D system of

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the x–y coupled rms envelope equations, 1.5D system will lose one degree of freedom, which means loss of possibly unstable modes and coupling effects. However, the basic idea and physics understanding are the same. Another advantage with 1.5D analysis is that it is easier to obtain the *Poincaré* surface plot to show the nonlinear phenomena. In the 1.5D system discussed in the paper, the envelope tune is defined as $\nu = \phi/360$, where ϕ is the eigenphase of the *Poincaré* tangent map $T(s)$ in one focusing period at the SFP. It is proved that the process of the SFP periodic-two bifurcation, representing the matched beam, is exactly related to the collective envelope instability stop band, also named as half-integer resonance or parametric resonance, where the matched envelope tune $\nu = 1/2$. Whereas, other nth order resonances $\nu_{l,n} = l/n$ $(n > 2)$, coming from periodic-n bifurcation only lead to mismatched resonance behavior around the SFP but no collective instability at all because a new SFP emerges immediately. The resonance conditions and instability regions are precisely predicted by the envelope tune $\nu_{l,n}$. For specific cases, regular orbits, resonance islands, and chaotic regions formed by resonance overlapping are clearly depicted with frequency analysis and Lyapunov spectral exponents.

This paper is organized as follows. In Section 2, the general Hamiltonian equation for envelope oscillation in a periodic channel is presented. For symmetric beam, we limited our research on the invariant manifold that leads to a 1.5D time dependent nonautonomous system. In Section 3, the nonlinear resonance properties are studied with classic perturbation theory. The mechanism of envelope instability, which actually lies in the process of periodic-two bifurcation, is discussed in detail. In Section 4, the conditions when higher orders of resonances take place and the phenomenon of period-n bifurcation are discussed. The frequency analysis together with the Lyapunov spectral exponents gives precise locations of chaotic region, resonance islands and regular orbits for when beam evolve under specific conditions. Discussion and conclusions are given in Section 5.

2. The Hamiltonian for the envelope phase space of symmetric beam

Using the longitudinal distance s as the time coordinate, generally, the KV Hamiltonian H_{env} for a transport channel can be expressed as

$$H_{env} = \frac{1}{2}\left(p_x^2 + p_y^2 + k_x(s)\tilde{x}^2 + k_y(s)\tilde{y}^2\right) - 1/2K\,\ln(\tilde{x}+\tilde{y}) + \frac{\tilde{\varepsilon_x}^2}{2\tilde{x}^2} + \frac{\tilde{\varepsilon_y}^2}{2\tilde{y}^2}. \tag{1}$$

Here, $\tilde{x}(s) = \sqrt{\overline{x^2(s)}}$ and $\tilde{y}(s) = \sqrt{\overline{y^2(s)}}$ are considered as the rms size of beam in real space, $\tilde{\varepsilon_x}(s) = \sqrt{\overline{x^2}\,\overline{p_x^2} - (\overline{xp_x})^2}$ and $\tilde{\varepsilon_y}(s) = \sqrt{\overline{y^2}\,\overline{p_y^2} - (\overline{yp_y})^2}$ are rms beam emittance in (x, p_x) and (y, p_y) planes, where $\overline{\cdots}$ denotes a statistical average over the beam space, $K = 2eI/4\pi\varepsilon_0 mc^3\beta^3\gamma^3$ is defined as generalized perveance, which is a dimensionless measure for the strength of the space charge, $k_x(s)$ and $k_y(s)$ represent the external focusing field function with the lattice period S, which satisfy $k_x(s) = k_x(s+nS)$ and $k_y(s) = k_y(s+nS)$, where $n = \pm 1, \pm 2, \ldots$ is the number of periods. In the paraxial symmetric solenoid focusing channel, $k_x(s) = k_y(s) = k(s) = [qB_z(s)/2m\gamma_b\beta_b c]^2$ where $B_z(s)$ is solenoid field strength. Assuming $\tilde{x} = \tilde{y} = r$, the motion of equation could be normalized as

$$\frac{d^2r}{ds^2} + k_z(s)r - \frac{K}{r} - \frac{1}{r^3} = 0. \tag{2}$$

For a more general discussion, the normalized axial magnetic field profile $k_z(s) = k_z(s+1)$ is expressed with the Fourier expansions. In the following, we apply the method to an example with

$$k_z(s) = (a_0 + a_1\cos(2\pi s))^2 \tag{3}$$

as suggested in Ref. [11–13]. Before discussion, we introduce several terminologies used in this paper.

Matched beam: Noted as $r_m(s)$, possible solution of Eq. (2), which is perfectly repeated from period to period. In the *Poincaré* section of plot, it is represented by the unique SFP. If beam leaves the SFP, then it is called mismatched.

Phase advance σ: In the matched condition, the phase advance is defined as the phase shift of a particle under the influence of space charge defocusing and external focusing over one period. It can be evaluated as

$$\sigma = \sigma(K) = \int_s^{s+1} \frac{\tilde{\varepsilon}}{r_m(s)^2} ds \tag{4}$$

In the limit of zero space charge $K = 0$, $\sigma(0) = \sigma_0$ is noted as the undepressed zero beam current phase advance.

Tune depression η: $\eta = \sigma/\sigma_0$ is defined as tune depression to measure the importance of the space charge. Fixed parameters (a_0, a_1, K) determined (σ_0, η) uniquely.

3. Envelope instability: period two orbits bifurcation of matched SFP

Following the former study on beam envelope instability [7,9,10,12–14,20], the eigenvalues $\lambda = |\lambda| e^{i\phi}$ of the Poincaré tangent map $T(s)$ in one focusing period at matched SFP indicate the growth rate $(|\lambda|)$ and phase shift (ϕ) of the perturbation of the matched envelope oscillation in one period [21]. Regions with $\lambda \neq 1$, named as envelope instability stop band, represent that the beam envelope cannot tolerate any tiny perturbation and will increase exponentially. In the following discussion, we do not distinguish between the terminology "envelope instability" and "matched SFP instability". With similar method as in Ref. [20], the stability characteristics of the nonautonomous system equation (2) are studied. Fig. 1 gives an example, with $a0 = 1.7$, $a1 = 1.07$, of the eigenvalues and eigenphases of the tangent map $T(s)$ at the matched SFP as function of phase advance σ, which decreases from σ_0 as the space charge term K increases.

Firstly, it can be noted that, for a specific space charge parameter K, there are 3 eigenvalues noted as λ_1, λ_2, λ_3 corresponding to the direction of time s, position r and momentum p_r. Only one collective mode, analogous to "breathing mode" or "quadrupole mode" in 2.5D system, could be excited. λ_1 refers to the independent variable time with the condition $\log_e|\lambda_1| = 1$ and $\phi_1 = 0$. In the direction r and p_r, the eigenvalues λ_2 and λ_3 are reciprocal pairs and complex conjugates which satisfy $|\lambda_2| = 1/|\lambda_3^*|$ and $\phi_2 = \phi_3^*$, which means the total phase space of the Hamiltonian system meets the Liouville theorem. If the phase space in one direction is stretched, the other is always compressed. Secondly, there is one envelope instability stop band between the region $53.5^\circ \le \sigma \le 70^\circ$ $(1.84 \le K \le 3.05)$ with $|\lambda_2| = 1/|\lambda_3^*| \neq 1$. Not surprisingly, this collective instability is excited whenever the condition $\phi_2 = \phi_3 = 180^\circ$ is satisfied, which is termed as half integer resonance with oscillation tune $\nu = 180/360 = 1/2$.

Fig. 2(a)–(f) shows the topology configuration evolution of the (r, p_r) phase space. With zero beam current $(K = 0)$, Fig. 2(a), the unique SFP represents the matched beam, which is surrounded by an infinite number of invariant tori, each describing a mismatched beam whose envelope exhibits stable betatron oscillation around the matched SFP. At the point where increasing space charge causes the eigenvalues to leave the axis, the unique SFP disappears and bifurcates into period two orbits, Fig. 2(b)–(d), where $K = 1.9$, 2.2, 3 respectively; the original SFP turns into a *saddle−node*, and