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Size-dependent torsion of functionally graded bars

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ABSTRACT

In this paper, size-dependent static and dynamic behavior of functionally graded microbars is investigated on the basis of the modified couple stress theory. The equation of motion and corresponding boundary conditions are derived using Hamilton's principle and presented in the dimensionless form. Equivalent mechanical properties (i.e. shear modulus, density and length scale) are extracted for the functionally graded microbar based on the mechanical properties of the material constituents. In this work, it is shown that without any simplifying assumption, two equivalent length scale parameters can be defined for functionally graded bars and the size-dependent mechanical behavior of these components can be explained using these parameters. As an example, static and dynamic behavior of a functionally graded microbar with fixed-free boundary conditions is analyzed and the effect of size-dependency on mechanical behavior of this structure is discussed.

1. Introduction

Functionally graded materials (FGMs) are inhomogeneous composites which are made of two different materials typically a metal and a ceramic. The volume fractions of material constituents in FGMs vary continuously in a certain direction of the structure. This continuous variation provides a smooth change in the mechanical properties and eliminates the high magnitude shear stresses which appear in laminated composites. FGMs can offer the benefits of both of material constituents and hence, these materials are widely used in various fields of engineering including aerospace, bioengineering and mechanical engineering.

Many researchers have studied the mechanical behavior of structures made of functionally graded materials. Here some of these works are reviewed. Fallah et al. [1] investigated buckling and vibration of functionally graded Euler–Bernoulli beams under thermal loading. Nonlinear vibration of functionally graded beams is investigated by Ke et al. [2]. They considered the material properties to be varying through the thickness of the beam and took into account the nonlinearities caused by the mid-plane stretching. Sarkar and Ganguli [3] presented a closed form solution for natural frequency of functionally graded Timoshenko beams. The nonlinear mechanical response of functionally graded piezoelectric beams with properties varying through the thickness of the beam is analyzed by Lin and Muliana [4]. Su and Banerjee [5] utilized the dynamic stiffness method to investigate the dynamic behavior and natural frequency of functionally graded Timoshenko beams. Torsional responses of functionally graded bars and cylinders are investigated by Horgan and Chan [6] and Dung and Hoa [7,8] respectively. Rahaeifard et al. showed that the sensitivity of atomic force microscopes can be enhanced using functionally graded materials [9,10]. A closed form solution is proposed for static response of functionally graded Kirchhoff plates by Apuzzo et al. [11]. Hosseini-Hashemia et al. investigated the in-plane and out of plane vibration of functionally graded rectangular plates with simply supported boundary conditions [12]. Furthermore, mechanical behavior of functionally graded shells has been widely investigated by researchers (see for example [13–17]).

Nowadays, FGMs are also used in microsystems such as thin films in the form of shape memory alloys [18,19] and micro-electromechanical systems [18,20]. In these systems, the size of structures is of order of microns and sub-microns. Many experimental researches performed on the mechanical response of micro scale structures (see for example [21–24]) proved that the mechanical behavior of these components is different from that predicted by the classical continuum theory. According to these researches, the normalized stiffness and normalized deflection of micro scale structures, which according to the classical theory should be independent of the structure size, is significantly size-dependent. Furthermore, the stiffness of micro scale structures measured in

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experiment, is much more than the stiffness predicted by the classical theory.

These experiments clearly showed that the classical continuum theory can neither evaluate the mechanical response of micro scale structures nor justify the size-dependency observed in mechanical behavior of these components. In contrast to the classical theory, some new continuum theories such as couple stress theory and modified couple stress theory can cover this gap and accurately model the mechanical behavior of micro scale structures.

The modified couple stress theory was developed by Yang et al. [25] by performing a modification on the couple stress theory previously proposed by Toupin [26] and also Mindlin and Tiersten [27]. In the modified couple stress theory, beside the classical stresses (i.e. normal and shear stresses), a non-classical stress known as couple stress is also acting on an element of material. This higher order stress is related to the displacement field by a new material constant called length scale parameter. Furthermore, in this theory, in addition to the classical equilibrium equations, a non-classic equilibrium equation (i.e. the equilibrium equation of moment of couples) is satisfied.

Many researchers utilized this theory to investigate the mechanical behavior of micro scale structures as well as to justify the size-dependency observed in experiment. Park and Gao [28] utilized this theory to derive the governing equations of Euler–Bernoulli microbeams and detected the size-dependency in the static behavior of these structures. Tsiatas [29] utilized this theory to investigate the mechanical behavior of Kirchhoff microplates with arbitrary shapes. Size-dependent static and dynamic behavior of electrostatically actuated microcantilevers is analyzed by Rahaeifard et al. [30,31]. They showed that utilizing the modified couple stress theory can remove the gap between the experimental and theoretical results of the static pull-in of microcantilevers [30]. They also proposed a non-classical yield criterion based on the modified couple stress theory to justify the size-dependent yielding of micro scale structures. The predictions of their criterion were in very good agreement with experimental observations while the yielding load predicted by the classical theory was up to 50% less than the yielding load measured in experiment [32].

This theory is also widely used by researchers to model the static and dynamic behavior of micro scale structures made of functionally graded materials. The size-dependent formulations of functionally graded Euler–Bernoulli and Timoshenko microbeams are developed by Asghari et al. [33,34]. They utilized this formulation to model the size-dependent deflection and natural frequency of these components. The size-dependent mechanical behavior of functionally graded Mindlin microplates is investigated by Ke et al. [35]. Thai and Kim [36] derived the governing equations of motion for functionally graded Reddy plates based on the modified couple stress theory and investigated the static and dynamic behavior of these structures.

In this paper based on the modified couple stress theory, a formulation is developed to model the static and dynamic torsion of functionally graded microbars. The volume fraction of material constituents is assumed to vary through the radial direction. On this basis, two equivalent length scale parameters are defined for functionally graded microbars based on the length scale parameters and volume fractions of material constituents. As a case study, static deformation and free vibration of a functionally graded fixed-free bar is analyzed. It is shown that the modified couple stress theory can successfully model the size-dependency of mechanical behavior of functionally graded bars. The results of the modified couple stress theory are compared with those evaluated based on the classical theory and the importance of size effect in mechanical behavior of micro scale functionally graded bars is discussed.

2. Preliminaries

According to the modified couple stress theory presented by Yang et al. [18], the strain energy of an elastic body can be written as follows.

$$U=\frac{1}{2}\int_V\left(\sigma_{ij}\varepsilon_{ij}+m_{ij}\chi_{ij}\right)dV \quad (i,j=1,2,3), \tag{1}$$

in which σ_{ij} and ε_{ij}, denote the components of stress and strain tensors and m_{ij} and χ_{ij} refer to the deviatoric part of couple stress and the symmetric curvature tensors respectively. These parameters can be expressed as

$$\varepsilon_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right), \tag{2}$$

$$\sigma_{ij}=\frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij}+2\mu\varepsilon_{ij}, \tag{3}$$

$$\chi_{ij}=\frac{1}{2}\left(\theta_{i,j}+\theta_{j,i}\right), \tag{4}$$

$$m_{ij}=2l^2\mu\chi_{ij}, \tag{5}$$

where u_i denotes the components of displacement field, l is the length scale parameter, E and μ denote the Yong modulus and shear modulus respectively and θ_i refers to the components of the rotation vector related to the components of the displacement field as follows.

$$\theta_i=\frac{1}{2}(curl(\mathbf{u}))_i. \tag{6}$$

3. Modelling of functionally graded bar

Fig. 1 displays a functionally graded microbar of length L having a circular cross section with radius R and area A. The microbar is under distributed torque (torque per unit length) denoted by T^d.

For this microbar the displacement field can be written as [37,38].

$$u_1=-y\varphi(z,t),\ \ u_2=x\varphi(z,t),\ \ u_3(z,t)=0, \tag{7}$$

where u_1, u_2 and u_3 refer to displacements along x, y, and z axes respectively and φ denotes the rotation angle of the cross section about z axis. According to the abovementioned displacement field, the components of the strain and symmetric curvature tensors can be calculated using Equations (2) and (4) as follows.

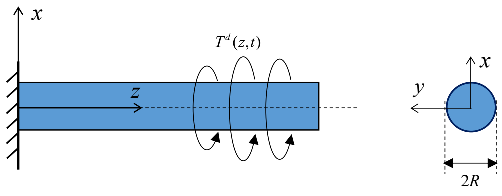

Fig. 1. A functionally graded circular microbar.