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# Addendum to “Spin decoherence rate in a homogenous all-electric ring”

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ABSTRACT

A recent paper by the author [1] derived analytical formulas for the spin decoherence rate for spin-polarized beams in models of all-electric storage rings. This paper presents additional results for the spin decoherence rate, due to vertical betatron oscillations. Contact is made with the work of other authors on the subject.



## 1. Introduction

A recent paper by the author [1] derived analytical formulas for the spin coherence rate for spin-polarized beams in a homogeneous weak focusing electrostatic storage ring. As stated in [1], calculations of the spin decoherence rate are of interest for the design and analysis of experiments to detect a permanent electric dipole moment (EDM) of spin-polarized particles circulating in storage rings. Such formulas can serve to provide useful benchmark tests for numerical simulations of proposed EDM experiments. The formulas can also provide insights in their own right, for the behavior of the beam polarization in various scenarios. This paper presents additional results to those in [1], to derive formulas for the spin coherence rate due to vertical betatron oscillations. Contact is made with the results in [2,3], where formulas for the spin coherence rate due to vertical betatron oscillations in a homogeneous weak focusing electrostatic storage ring were also derived. The results presented here agree with those in [2,3], but are derived under more general conditions than the assumptions in [2,3].

The structure of this paper is as follows. The basic formalism is presented in Section 2. Section 3 presents the analysis for bounded vertical oscillations while Section 4 treats the case of no vertical focusing. The spin decoherence rate is analyzed in Section 5. Section 6 presents a brief comment on some statements in [4]. Section 7 concludes.

## 2. Basic formalism

Following [1], I treat a particle of mass $m$ and charge $e$, with velocity $\boldsymbol{v}=\boldsymbol{\beta}c$, Lorentz factor $\gamma=1/\sqrt{1-\beta^2}$, position vector $\boldsymbol{r}$ and momentum $\boldsymbol{p}$. I shall set $c=1$ below. The particle spin $\boldsymbol{s}$ is treated as a unit vector and $a=\frac{1}{2}(g-2)$ denotes the magnetic moment anomaly. I treat an all-electric model, hence $\boldsymbol{p}=\gamma m\boldsymbol{v}$. The reference orbit is a circle of radius $r_0$. The independent variable is the azimuth $\theta$ along the reference orbit. The transverse coordinates are $(x,z)$ which are radial and vertical, respectively, and the radius is $r=r_0+x$. The Hamiltonian for the orbital motion in a homogeneous weak focusing all-electric ring is

$$K=-r\left[(H-e\Phi)^2-m^2-p_x^2-p_z^2\right]^{1/2}. \quad (1)$$

Here $H$ is the total energy, $\Phi(r,z)$ is the electrostatic potential and the electric field is $\boldsymbol{E}=-\nabla\Phi$. We define $\Phi=0$ on the reference orbit. Note that $H=\gamma mc^2+e\Phi$. The values of parameters on the reference orbit are denoted by a subscript “0” hence the energy and momentum of the reference particle are $H_0$ and $p_0$, respectively. The electric field gradient is parameterized by the field index $n$, where $E_x\propto 1/r^{1+n}$ for $z=0$. For the case $n=0$ (no vertical focusing), the potential is logarithmic $e\Phi=m\gamma_0\beta_0^2\ln(r/r_0)$. The analysis below treats bounded vertical oscillations, where the field index is $n>0$. The potential is given by a hypergeometric function. To the required order

$$e\Phi=\frac{m\gamma_0\beta_0^2}{n}\left\{1-\frac{r_0^n}{r^n}\,{}_2F_1\left(\frac{n}{2},\frac{n}{2};\frac{1}{2};-\frac{z^2}{r^2}\right)\right\}$$

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$$= \frac{p_0\beta_0}{n}\left\{1-\frac{r_0^n}{r^n}\left[1-\frac{n^2}{2!}\frac{z^2}{r^2}+\cdots\right]\right\}$$
$$\simeq p_0\beta_0\left[\frac{x}{r_0}-\frac{n+1}{2}\frac{x^2}{r_0^2}+\frac{n}{2}\frac{z^2}{r_0^2}\left(1-(n+2)\frac{x}{r_0}\right)\right]. \tag{2}$$

Synchrotron oscillations and rf cavities will be treated later, but ignored for now. Hence for now the total energy $H$ is a dynamical invariant. The orbital equations of motion in the transverse plane are

$$\frac{dx}{d\theta}=\frac{\partial K}{\partial p_x}=-r^2\frac{p_x}{K} \tag{3a}$$

$$\frac{dz}{d\theta}=\frac{\partial K}{\partial p_z}=-r^2\frac{p_z}{K} \tag{3b}$$

$$\frac{dp_x}{d\theta}=-\frac{\partial K}{\partial x}=r^2\frac{H-e\Phi}{K}\frac{\partial(e\Phi)}{\partial x}-\frac{K}{r} \tag{3c}$$

$$\frac{dp_z}{d\theta}=-\frac{\partial K}{\partial z}=r^2\frac{H-e\Phi}{K}\frac{\partial(e\Phi)}{\partial z}. \tag{3d}$$

I set $H=H_0(1+\Delta H/H_0)$. Later I shall treat synchrotron oscillations, but for now $H$ is invariant and so $\Delta H/H_0$ is constant.

## 3. Vertical oscillations

In [1], the spin decoherence rate was calculated for numerous scenarios, mostly involving horizontal orbital motion. In this addendum, I treat vertical betatron oscillations. The specific model is that of free (bounded) vertical betatron oscillations. Because of (small) transverse coupling terms in Eq. (3), of $O(z^2)$ and $O(p_z^2)$, the vertical betatron oscillations drive (small amplitude) horizontal betatron oscillations. (There are no free oscillations in the horizontal plane.) Then to linear order,

$$\frac{d(z/r_0)}{d\theta}=\frac{p_z}{p_0} \tag{4a}$$

$$\frac{d(p_z/p_0)}{d\theta}=-\frac{r_0^2}{p_0}\frac{H_0}{p_0r_0}H_0\beta_0^2\frac{nz}{r_0^2}=-n\frac{z}{r_0} \tag{4b}$$

$$\frac{d^2}{d\theta^2}\frac{z}{r_0}=-n\frac{z}{r_0}. \tag{4c}$$

It is well known that the small amplitude vertical betatron tune is $\nu_z=\sqrt{n}$. We parameterize the vertical betatron oscillations using an amplitude parameter $z'_0$ and an initial phase $\phi_{z0}$:

$$\frac{z}{r_0}=\frac{z'_0}{\nu_z}\sin(\nu_z\theta+\phi_{z0}), \tag{5a}$$

$$z'\equiv\frac{p_z}{p_0}=z'_0\cos(\nu_z\theta+\phi_{z0}). \tag{5b}$$

Next we treat the horizontal motion. Note that the driven oscillations are of the second order in small quantities, i.e. $x=O(z_0'^2)$ and $p_x=O(z_0'^2)$. Also $\Delta H/H_0$ is restricted to be of the second order in small quantities. The following expression is required for the derivation below:

$$\frac{K^2}{r^2}=(H_0+\Delta H-e\Phi)^2-m^2-p_x^2-p_z^2\simeq p_0^2+2H_0(\Delta H-e\Phi)$$
$$-p_z^2\simeq p_0^2-p_z^2+2p_0^2\left(\frac{\Delta H}{H_0\beta_0^2}-\frac{x}{r_0}-\frac{n}{2}\frac{z^2}{r_0^2}\right). \tag{6}$$

Then $dx/d\theta\simeq p_x/p_0$ and

$$\frac{d}{d\theta}\frac{p_x}{p_0}\simeq\frac{r}{Kp_0}\left[r(H-e\Phi)\frac{\partial(e\Phi)}{\partial x}-\frac{K^2}{r^2}\right]$$
$$\simeq-\left[\left(1+\frac{x}{r_0}\right)\left(1+\frac{\Delta H}{H_0}-\beta_0^2\left(\frac{x}{r_0}+\frac{n}{2}\frac{z^2}{r_0^2}\right)\right)\right.$$
$$\times\left(1-(n+1)\frac{x}{r_0}-\frac{n(n+2)}{2}\frac{z^2}{r_0^2}\right)-1$$
$$\left.+\frac{p_z^2}{p_0^2}-2\frac{\Delta H}{H_0\beta_0^2}+\frac{2x}{r_0}+n\frac{z^2}{r_0^2}\right]=-(2-\beta_0^2-n)\frac{x}{r_0}$$
$$+\frac{n(n+\beta_0^2)}{2}\frac{z^2}{r_0^2}-\frac{p_z^2}{p_0^2}+\frac{2-\beta_0^2}{\beta_0^2}\frac{\Delta H}{H_0}. \tag{7}$$

It is well known that the small amplitude horizontal betatron tune is $\nu_x=\sqrt{2-\beta_0^2-n}$. Since the radial motion consists of bounded oscillations, one must have $\langle dp_x/d\theta\rangle=0$. We use the result $\langle(p_z/p_0)^2\rangle=\nu_z^2\langle(z/r_0))^2\rangle=n\langle(z/r_0)^2\rangle$ below to obtain

$$(2-\beta_0^2-n)\left\langle\frac{x}{r_0}\right\rangle=\frac{n(n+\beta_0^2)}{2}\left\langle\frac{z^2}{r_0^2}\right\rangle-\left\langle\frac{p_z^2}{p_0^2}\right\rangle$$
$$+\frac{2-\beta_0^2}{\beta_0^2}\frac{\Delta H}{H_0}=-\frac{2-\beta_0^2-n}{2}\left\langle\frac{p_z^2}{p_0^2}\right\rangle+\frac{2-\beta_0^2}{\beta_0^2}\frac{\Delta H}{H_0}. \tag{8}$$

Hence

$$\left\langle\frac{x}{r_0}\right\rangle=-\frac{1}{2}\left\langle\frac{p_z^2}{p_0^2}\right\rangle+\frac{2-\beta_0^2}{\nu_x^2\beta_0^2}\frac{\Delta H}{H_0}. \tag{9}$$

Note that it was derived in [5] that the dispersion in this model is given by $D_x=r_0(2-\beta_0^2)/\nu_x^2$, so the above expression can be written in the more informative way:

$$\left\langle\frac{x}{r_0}\right\rangle=-\frac{1}{2}\left\langle\frac{p_z^2}{p_0^2}\right\rangle+\frac{1}{\beta_0^2}\frac{D_x}{r_0}\frac{\Delta H}{H_0}. \tag{10}$$

The second term is simply the usual dispersion contribution to the radial displacement. The first term is a driving term from the vertical betatron oscillations. Continuing with the analysis,

$$\left\langle\frac{e\Phi}{H_0}\right\rangle\simeq\beta_0^2\left(\left\langle\frac{x}{r_0}\right\rangle+\frac{n}{2}\left\langle\frac{z^2}{r_0^2}\right\rangle\right)=\frac{2-\beta_0^2}{\nu_x^2}\frac{\Delta H}{H_0}. \tag{11}$$

It follows that

$$\left\langle\frac{\Delta\gamma}{\gamma_0}\right\rangle=\left\langle\frac{H-e\Phi}{H_0}\right\rangle-1=\left(1-\frac{2-\beta_0^2}{2-\beta_0^2-n}\right)\frac{\Delta H}{H_0}=-\frac{n}{\nu_x^2}\frac{\Delta H}{H_0}. \tag{12}$$

Specializing to the case of on-energy motion $\Delta H/H_0=0$ and using $z'_0$ yield

$$\left\langle\frac{\Delta\gamma}{\gamma_0}\right\rangle=0, \tag{13}$$

$$\left\langle\frac{x}{r_0}\right\rangle=-\frac{1}{2}\left\langle\frac{p_z^2}{p_0^2}\right\rangle=-\frac{\langle z_0'^2\rangle}{2}. \tag{14}$$

The above expressions agree with Orlov's results derived in [2] and [3], respectively. It is also easily verified that $\langle(\Delta\gamma/\gamma_0)^2\rangle=O(z_0'^4)$.

The derivations of Eqs. (13) and (14) in [2] and [3] respectively were made under the approximation of very weak vertical focusing $0<n\lll 1$. (This is also stated in [4].) The derivation in this paper shows that such an approximation is unnecessary: *no* restriction was required on the field index (other than $n>0$, so as to have bounded vertical oscillations). Tracking simulations confirm that Eqs. (13) and (14) are valid for arbitrary values $0<n<1$. There may be a caveat that the value of $\nu_z$ should not be rational, to avoid orbital resonances, to justify the statistical averages.

It is stated in [2], that Eq. (13) is valid in the presence of rf and synchrotron oscillations (and by extension Eq. (14) also). We can derive this as follows. Define $\tau=t-t_*$ where $t_*=(r_0/v_0)\theta$ is the