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"Influence Method". Detailed mathematical description

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ABSTRACT

A new method for the absolute determination of nuclear particle flux in the absence of known detector efficiency, the "Influence Method", was recently published (I.J. Rios and R.E. Mayer, Nuclear Instruments & Methods in Physics Research A 775 (2015) 99–104). The method defines an estimator for the population and another estimator for the efficiency.

In this article we present a detailed mathematical description which yields the conditions for its application, the probability distributions of the estimators and their characteristic parameters. An analysis of the different cases leads to expressions of the estimators and their uncertainties.

1. Introduction

A new method for the absolute determination of particle flux in the absence of known detector efficiency "Influence Method" was recently published [1]. This method exploits the influence of the presence of one detector, in the count rate of another detector. This influence is expressed as a modification in the detection probability of a second detector after the radiation has traversed the first detector, allowing to derive a statistical estimator for the absolute number of incident particles, independent of the efficiency of the detectors. Another estimator is deduced for the detection efficiency, calculated from the same experiment.

It is applicable to any source subject to the limitation of constant efficiency or a monoenergetic source. In the particular case of time-of-flight spectrometry, widely employed in pulsed neutron sources spectrum determinations, the estimators could be applied to each time bin, thus rendering the particle energy spectrum in absolute terms and, the energy dependent efficiency of the detector system.

Let, in the simplest case, two detectors with the same efficiency ε, be placed one behind the other at a certain distance from the radiation source as schematized in Fig. 1. The number of particles counted by detector A is an aleatory variable (X) whose distribution is a binomial of parameters n and ε ($X \sim Bi(n,\varepsilon)$). In the proposed scheme, particles not detected at A ($X_{out} = n - X$) impinge on detector B. Thus, the number of those particles detected by B are an aleatory variable (Y) whose distribution is also a binomial of parameters n and $\varepsilon \cdot (1-\varepsilon) = \varepsilon \cdot q$ (demonstration that $Y \sim Bi(n, \varepsilon q)$ is shown in Appendix A), where $q = (1-\varepsilon)$ represents the probability of not being detected by A.

This scheme can be interpreted as a method where the sample of the second variable is influenced by the first one, for which reason we call it the "Influence Method". This influence manifests itself through the correlation between X and Y which ends up determined by ε (an analysis of the bivariate binomial distribution can be found in [2]).

Within this scheme we define an estimator for the population

$$\hat{n} = \frac{X^2}{X - Y} \tag{1}$$

and an estimator for the efficiency as:##

$$\hat{\varepsilon} = \frac{X - Y}{X} \tag{2}$$

In the practical case, it is very important for the right application of Eqs. (1) and (2), that X and Y come from the same source of particles and that both detectors be protected from other spurious sources that could affect the counting (background subtracted).

Given the joint probability mass function for both variables in the Influence Method, it will be shown that the proposed estimator for the population $\hat{n} = X^2/(X-Y)$ represents the trajectory of the maximum of the distribution in the XY plane for any ε.

2. Joint probability distribution

Given the fact that the distributions are $X \sim Bi(n,\varepsilon)$, $Y \sim Bi(n-X,\varepsilon)$ the joint probability mass function is

$$P(X=x, Y=y) = P(Y=y \mid X=x) \cdot P(X=x) = \binom{(n-x)}{y} \varepsilon^y (1-\varepsilon)^{(n-x)-y}$$

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$$\times\binom{n}{x}\varepsilon^{x}(1-\varepsilon)^{n-x}=\binom{(n-x)}{y}\binom{n}{x}\varepsilon^{y+x}(1-\varepsilon)^{2(n-x)-y} \tag{3}$$

For $y \le (n-x)$, and zero for the rest. In Fig. 2 depicts a superposition of the joint probability for $\varepsilon=0.2$, $\varepsilon=0.4$, $\varepsilon=0.6$ and $\varepsilon=0.8$ shown in the $(x/n,\ y/n)$ plane for normalization reasons.

The expected value of the variable $X \sim Bi(n,\varepsilon)$ is

$$\mu_x = n\varepsilon \tag{4}$$

And its variance

$$\sigma_x{}^2 = n\varepsilon q = n\varepsilon(1-\varepsilon) \tag{5}$$

Appendix A contains the demonstration that, given that $Y \sim Bi(n-X,\varepsilon)$, in the proposed scheme its distribution can be written as $Y \sim Bi(n,\varepsilon q)$ and, consequently, its expected value is

$$\mu_y = n\varepsilon q = n\varepsilon(1-\varepsilon) \tag{6}$$

Its variance will be

$$\sigma_y{}^2 = n\varepsilon q(1-\varepsilon q) = n\varepsilon(1-\varepsilon)(1-\varepsilon+\varepsilon^2) \tag{7}$$

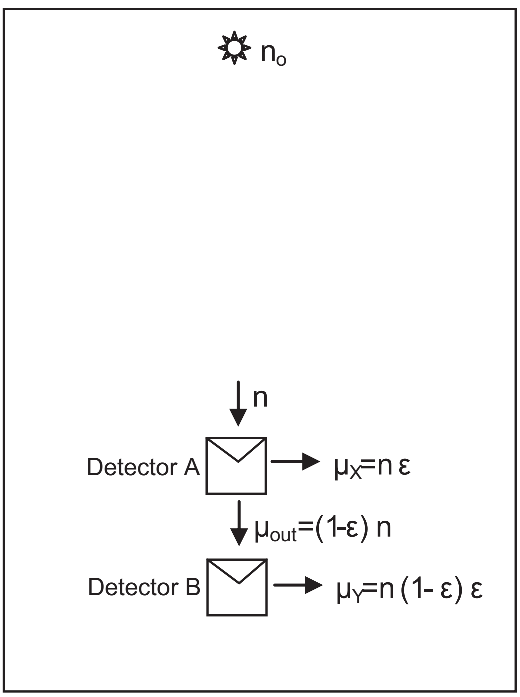

Fig. 1. Scheme of the measurement array proposed by the "Influence Method".

Its covariance σ_{xy} and the correlation coefficient ρ between these variables are deduced in Appendix B.

Observing the expression of the correlation coefficient (Eq. B.8) it can be seen that if the success probability, that is detection, is low $(\varepsilon \to 0)$ then $\rho \to 0$ and so they are practically independent, while when ε is high $(\varepsilon \to 1)$ then $\rho \to -1$ and they are highly correlated.

The most probable value of the joint distribution is found at the position $(x=\mu_x, y=\mu_y)$ in the XY plane. But it can be seen (using Eqs. (4) and (6)) that $\mu_y = \mu_x - \mu_x^2/n$ and thus $y = x - x^2/n$ represents the trajectory of the most probable value of the joint distribution in the $(X,\ Y)$ plane (dotted line in Fig. 2). Then, we define the population estimator by Eq. (1). Also $\mu_y = \mu_x(1-\varepsilon)$ and so, the corresponding estimator for the success probability is defined

by Eq. (2).

3. Mean value of the estimators

Given the fact that the estimator $\hat{n}$ is a nonlinear function of two dependent variables X and Y, its mean value can be found taking the Taylor expansion of the function around the mean value [3,4]. In this manner, the mathematical expectation of a function $G = g(x,\ y)$ can be approximated as:

$$E(G) \cong g(\mu_x, \mu_y) + \left(\frac{\partial^2 G}{\partial x^2}\right)\frac{\sigma_x{}^2}{2} + \left(\frac{\partial^2 G}{\partial y^2}\right)\frac{\sigma_y{}^2}{2} + \left(\frac{\partial^2 G}{\partial x \partial y}\right)\sigma_{xy} \tag{8}$$

The derivatives are evaluated at point (μ_x, μ_y).

The derivatives for estimator $\hat{n}$ (Eq. (1)) are

$$\begin{aligned}
&\frac{\partial \hat{n}}{\partial x} = \frac{x^2-2xy}{(x-y)^2} \qquad \frac{\partial^2 \hat{n}}{\partial x^2} = 2\frac{y^2}{(x-y)^3}\\
&\frac{\partial \hat{n}}{\partial y} = \frac{x^2}{(x-y)^2} \qquad \frac{\partial^2 \hat{n}}{\partial y^2} = \frac{2x^2}{(x-y)^3}\\
&\frac{\partial \hat{n}}{\partial y \partial x} = \frac{-2xy}{(x-y)^3}
\end{aligned} \tag{9}$$

Then, evaluating these derivatives at mean values (μ_x, μ_y) and placing them in Eq. (8), the expected value for the estimator $\hat{n}$ is

$$E(\hat{n}) \cong n + \left(\frac{\mu_y{}^2\sigma_x{}^2 + \mu_x{}^2\sigma_y{}^2 - 2\mu_x\mu_y\sigma_{xy}}{(\mu_x-\mu_y)^3}\right) \tag{10}$$

Thus, replacing Eq. (4) through (7) and (B.7) into (10)

$$E(\hat{n}) \cong n + \left(\frac{(1-\varepsilon)(2-\varepsilon)}{\varepsilon^3}\right) \tag{11}$$

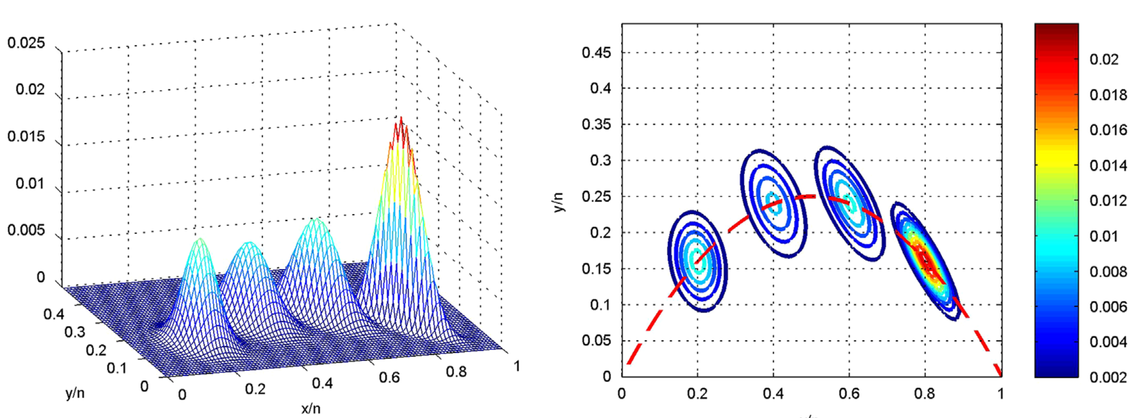

Fig. 2. Graph of the joint probability function for $\varepsilon=0.2$, $\varepsilon=0.4$, $\varepsilon=0.6$ and $\varepsilon=0.8$ shown in the $(x/n,\ y/n)$ plane for normalization reasons. In the right hand side the trajectory of the maximum is explicitly shown.