Composites Part B 75 (2015) 281–287

Contents lists available at ScienceDirect

Composites Part B

journal homepage: www.elsevier.com/locate/compositesb

# An exact solution for an anisotropic plate with an elliptic hole under arbitrary remote uniform moments

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ARTICLE INFO

*Article history:*
Received 7 October 2014
Received in revised form
5 December 2014
Accepted 2 February 2015
Available online 11 February 2015

*Keywords:*
B. Anisotropy
B. Elasticity
B. Stress concentrations
C. Analytical modelling

ABSTRACT

The problem of an anisotropic plate containing an elliptic hole subjected to remote bending or twisting moments is considered. In contrast with the previous works on the problem, the requirement that the deflection be a single-valued function is satisfied by introducing a correction constant. An exact solution for general anisotropic materials under arbitrary uniform loading conditions is derived. Explicit expressions for the deflection and moments on the edge of an elliptic hole in an orthotropic plate subjected to bending or twisting moments are obtained. The moment intensity factors as the elliptic hole degenerates into a crack are given.



## 1. Introduction

The problem of an elastic plate containing a hole subjected to a remote loading is important in engineering applications because of the stress concentrations at the edge of the hole [1,2]. The loading may include in-plane forces and out-of-plane moments. However, most of the studies on the hole problem are for in-plane forces (e.g., [3]) and relatively few results are available for bending or twisting moments. The present work is concerned with the latter. Specifically, an elliptic hole in a homogeneous anisotropic plate is considered. The size of the hole is assumed to be small compared with the over-all dimensions of the plate so that the plate may be regarded as infinitely large [4]. Moreover, the Kirchhoff plate theory is adopted.

Goodier [5] investigated the influence of circular and elliptic holes on the transverse flexure of isotropic elastic plates in various loading cases. Goodier's analyses were made by assuming the form of the deflection, hence no additional requirement on the deflection was needed. Lekhnitskii [6] developed a complex variable method for anisotropic plates and provided explicit solutions for a circular hole in an orthotropic plate. Lekhnitskii's treatment essentially assumed the form of derivatives of the deflection. Although Lekhnitskii mentioned that the requirement that deflection be a single-valued function should be imposed in the solution process, no detailed account was given. Subsequently Lekhnitskii's method was used to obtain moment distributions around holes in symmetric composite laminates subjected to bending moments by Prasad and Shuart [7] and by Ukadgaonker and Rao [8]. Based on Lekhnitskii's method, Hsieh and Hwu [9] developed a Stroh-like formalism and derived analytical solutions for anisotropic plates. The Stroh formalism is an elegant and powerful method for two-dimensional anisotropic elasticity [10]. A distinctive feature of the Stroh formalism is that the general solution is provided in terms of the eigenvalues and eigenvectors of an eigenvalue problem. By means of a similar Stroh-like formalism for coupled stretching and bending deformations, Cheng and Reddy [11] obtained a solution for a laminated anisotropic plate subjected to the remote uniform membrane stress resultants and bending moments. However, it appears that the aforementioned works, except [6], have not taken the requirement on the deflection into consideration. The validity of the solutions obtained is, therefore, limited to special types of material, shape of the hole and applied moment. It is the objective of this paper to derive a correct solution for general anisotropic plates under arbitrary uniform loading conditions with full consideration of the single-valuedness requirement.

The plan of the paper is as follows. The Stroh-like formalism proposed by Hsieh and Hwu [9] is first introduced. An exact solution corresponding to a single-valued deflection for general anisotropic materials is then derived using the Stroh-like

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http://dx.doi.org/10.1016/j.compositesb.2015.02.003
1359-8368/

formalism. Explicit expressions for the rotations, deflection and moments on the edge of a hole in an orthotropic material are obtained. The moment intensity factors as the elliptic hole degenerates into a crack are given. Finally some concluding remarks are made.

## 2. Stroh-like formalism

In the following discussions the Greek indices can be either 1 or 2 and summation over repeated indices is implied.

In the Kirchhoff plate theory, the displacements are assumed as

$$u_\alpha = x_3\theta_\alpha(x_1,x_2), \;\; u_3 = w(x_1,x_2), \tag{1}$$

where $u_\alpha$ are in-plane displacements, $u_3$ is deflection, and $\theta_\alpha$ are the rotations related to $w$ by

$$\theta_\alpha = -w_{,\alpha}. \tag{2}$$

The equilibrium equations are

$$M_{\alpha\beta,\alpha} = Q_\beta, Q_{\beta,\beta} = 0, \tag{3}$$

where $M_{\alpha\beta}$ are moments, $Q_\beta$ are shear forces, and a comma in the subscript stands for partial differentiation. The constitutive equations are

$$\begin{bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \kappa_{11} \\ \kappa_{22} \\ 2\kappa_{12} \end{bmatrix} \tag{4}$$

where $D_{ij}$, $i,j = 1,2,6$, are the bending stiffness constants and $\kappa_{\alpha\beta}$ are the curvatures defined by $\kappa_{\alpha\beta} = -w_{,\alpha\beta}$. Substitution of Eq. (4) into Eq. (3) leads to

$$D_{11}\frac{\partial^4 w}{\partial x_1^4} + 4D_{16}\frac{\partial^4 w}{\partial x_1^3 \partial x_2} + 2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + 4D_{26}\frac{\partial^4 w}{\partial x_1 \partial x_2^3} + D_{22}\frac{\partial^4 w}{\partial x_2^4} = 0. \tag{5}$$

The solution of $w$ is represented by Ref. [6]

$$w = 2Re[w_1(z_1) + w_2(z_2)], \tag{6}$$

where $z_\alpha = x_1 + \mu_\alpha x_2$ and $\mu_\alpha$ are the roots with positive imaginary part of the following equation:

$$D_{22}\mu^4 + 4D_{26}\mu^3 + 2(D_{12}+2D_{66})\mu^2 + 4D_{16}\mu + D_{11} = 0. \tag{7}$$

With Eqs. (6) and (2), the rotations can be expressed as

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = 2Re(\mathbf{Af}), \tag{8}$$

where $Re$ denotes the real part,

$$\mathbf{A} = -\begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix}, \;\; \mathbf{f} = \begin{bmatrix} w_1'(z_1) \\ w_2'(z_2) \end{bmatrix}, \tag{9}$$

and prime denotes the derivative with respect to the argument. With Eqs. (6) and (4), the moments $M_{11}$ and $M_{22}$ can be represented as

$$\begin{bmatrix} M_{11} \\ M_{22} \end{bmatrix} = 2Re\left(\begin{bmatrix} p_1 & p_2 \\ q_1 & q_2 \end{bmatrix} \begin{bmatrix} w_1'' \\ w_2'' \end{bmatrix}\right), \tag{10}$$

where

$$p_\alpha = D_{12}\mu_\alpha^2 + 2D_{16}\mu_\alpha + D_{11}, \;\; q_\alpha = D_{22}\mu_\alpha^2 + 2D_{26}\mu_\alpha + D_{12}. \tag{11}$$

From Eq. (3), the moments and shear forces can be expressed as

$$M_{11} = -\psi_{1,2}, \;\; M_{22} = \psi_{2,1}, \tag{12}$$

$$M_{12} = \psi_{1,1} - \eta = \eta - \psi_{2,2}, \tag{13}$$

$$Q_1 = -\eta_{,2}, Q_2 = \eta_{,1}, \tag{14}$$

where $\psi_1$, $\psi_2$ and $\eta$ are functions of $x_1$ and $x_2$. Using Eq. (12), general expressions for $\psi_1$ and $\psi_2$ are obtained from Eq. (10) as

$$\boldsymbol{\psi} = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = 2Re[\mathbf{Bf}], \tag{15}$$

where

$$\mathbf{B} = \begin{bmatrix} \dfrac{p_1}{\mu_1} & \dfrac{p_2}{\mu_2} \\ -q_1 & -q_2 \end{bmatrix}. \tag{16}$$

Let $\mathbf{s}$ and $\mathbf{n}$ be the tangential and normal unit vectors on the boundary of a plate such that

$$n_1 = -s_2, \;\; n_2 = s_1. \tag{17}$$

Eq. (14) yields

$$\frac{\partial \eta}{\partial s} = Q_1 n_1 + Q_2 n_2 = Q_n. \tag{18}$$

where $Q_n$ is the shear force on the boundary. Eqs. (12) and (13) give

$$\frac{\partial \boldsymbol{\psi}}{\partial s} = M_{nn}\mathbf{n} + (M_{sn} + \eta)\mathbf{s}, \tag{19}$$

$$\frac{\partial \boldsymbol{\psi}}{\partial n} = (\eta - M_{ns})\mathbf{n} - M_{ss}\mathbf{s}, \tag{20}$$

where

$$M_{nn} = \mathbf{n}^T\mathbf{Mn}, \;\; M_{sn} = \mathbf{s}^T\mathbf{Mn}, \;\; M_{ss} = \mathbf{s}^T\mathbf{Ms}. \tag{21}$$

Here

$$\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix} \tag{22}$$

Using Eqs. (19) and (20), the moments in Eq. (21) may also be expressed as

$$M_{nn} = \mathbf{n}^T\frac{\partial \boldsymbol{\psi}}{\partial s}, \;\; M_{sn} = \frac{1}{2}\left(\mathbf{s}^T\frac{\partial \boldsymbol{\psi}}{\partial s} - \mathbf{n}^T\frac{\partial \boldsymbol{\psi}}{\partial n}\right), \;\; M_{ss} = -\mathbf{s}^T\frac{\partial \boldsymbol{\psi}}{\partial n}. \tag{23}$$

Consider a coordinate system obtained by rotating an angle $\alpha$ about the $x_3$ axis and described by

$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{24}$$

With respect to the rotated coordinate system, a matrix equation for $\boldsymbol{\theta}$ and $\boldsymbol{\psi}$ is given by Ref. [11]