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# Closed form solutions for buckling and postbuckling analysis of imperfect laminated composite plates with piezoelectric actuators

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ABSTRACT

Buckling and postbuckling behavior of symmetric laminated composite plates with surface mounted and embedded piezoelectric actuators subjected to mechanical, thermal, electrical, and combined loads is studied. Formulation is based on the classical laminated plate theory with von-Karman non-linear kinematic relations. Initial geometrical imperfections are also accounted, and finally applying Galerkin procedure, the resulting equations are solved to obtain closed form expressions for non-linear equilibrium paths. Temperature dependency of thermo-mechanical properties is considered. Three cases of simply supported boundary conditions are investigated. Effects of in-plane compressive loading, temperature dependency and independency of properties, electrical loading, lay-up configuration, and geometric imperfection are discussed. Results for various states are verified with the known data in the literature.


## 1. Introduction

Nowadays the material sciences have intensive focus on the upgrade of strength and reinforcement of old materials based on the new necessities. Hence, due to extensive usage of fibrous composites in various industries, the improvement of quality of these materials is important, in which piezoelectric material is one of the ways to increase of durability of composites. Direct and converse piezoelectric effects have led to the development of adaptive piezolaminated composite materials and structures.

Compared to the several investigations on postbuckling behavior of laminated composite plates (see for example, [1–5]), there exist limited studies on postbuckling of piezolaminated plates. Oh et al. [6] presented non-linear finite element equations based on the layerwise plate theory, in which formulated for a piezolaminated plate subjected to thermal and piezoelectric loads. Also, they analyzed the postbuckling and vibration characteristics of square composite plate with fully and partially bonded piezoelectric actuators. Shen [7] studied thermal postbuckling of shear-deformable laminated plates with piezoelectric actuators. In that study, a mixed Galerkin perturbation technique was employed to determine thermal buckling loads and postbuckling equilibrium paths, and the material properties were assumed to be independent of the temperature. Also Shen [8] studied electro-thermo-mechanical buckling and postbuckling of shear-deformable laminated plates with piezoelectric actuators, using a mixed Galerkin perturbation technique, and temperature-independent material properties. Varelis and Saravanos [9] analyzed coupled multi-field generalized non-linear mechanics together with an associated plate finite element for the buckling and postbuckling response of active and sensory piezoelectric-composite laminated plates, which included non-linear effects due to large rotations and stress stiffening. In that work, coupled non-linear governing equations for piezolaminates were initially formulated, using mixed-field shear-layerwise kinematic assumptions. The discrete coupled equations of motion of the smart structure were finally linearized and solved using an incremental-iterative method based on the Newton–Raphson technique. Shen and Hong Zhu [10] extended the previous works to the case of shear deformable laminated plates with PFRC actuators under uniaxial compression, uniform temperature rise, and electric field component $E_z$. In that study, postbuckling analysis of laminated plates was based on the Reddy's higher order shear deformation plate theory with von Karman-type of kinematic non-linearity. This survey in the literature reveals that few investigations are reported on the postbuckling of laminated composite plates with piezoelectric actuators and sensors. Especially, the lack of a closed form solution for postbuckling of piezolaminated plates encouraged authors to develop closed form postbuckling analysis of other materials [11] for piezolaminated plates.

This paper presents closed form expressions to investigate buckling and postbuckling behaviors of symmetric laminated composite plates with surface mounted and embedded piezoelectric

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actuators under mechanical, thermal, electrical, and combined loads. Formulation is based on the classical laminated plate theory with von-Karman non-linear kinematic relations. Initial geometrical imperfections are also accounted, and finally applying Galerkin procedure, the resulting equations are solved to obtain closed form expressions for non-linear equilibrium paths. Temperature dependency of thermo-mechanical properties is considered. Three cases of boundary conditions are investigated. Effects of in-plane compressive loading, temperature dependency and independency of properties, electrical loading, lay-up configuration, and geometric imperfection are discussed. Results for various states are verified with the known data in the literature.

## 2. Theoretical formulation

Consider a rectangular plate made of perfectly bonded orthotropic composite plies and surface mounted and embedded six piezoelectric layers into laminated composites with total thickness $h$, as shown in Fig. 1. The length and width of the plate are $a$ and, $b$, respectively. Rectangular Cartesian coordinates $(x, y, z)$ are assumed for derivations and the mid-plane ($z = 0$) is a symmetrically plane of geometry and lay-up configuration. The angle of fibers in laminated composites is referenced with $x$ coordinates in this study.

According to the classical laminated plate theory, the displacement field is assumed to be [12]

$$\begin{aligned} u(x,y,z) &= u_0(x,y) - z w_{0,x}(x,y) \\ v(x,y,z) &= v_0(x,y) - z w_{0,y}(x,y) \\ w(x,y,z) &= w_0(x,y) \end{aligned} \tag{1}$$

here $(u, v, w)$ are the plate displacements parallel to the coordinates $(x, y, z)$, $(u_0, v_0, w_0)$ represent the displacements on the mid-plane ($z = 0$) of the plate, and a comma indicates the partial derivative.

The non-linear strain–displacement relations are [13,14]

$$\begin{aligned} \varepsilon_{xx} &= u_{,x} + \frac{1}{2} w_{,x}^2 + w_{,x}^* w_{,x} \\ \varepsilon_{yy} &= v_{,y} + \frac{1}{2} w_{,y}^2 + w_{,y} w_{,y}^* \\ \gamma_{xy} &= u_{,y} + v_{,x} + w_{,x} w_{,y} + w_{,x}^* w_{,y} + w_{,x} w_{,y}^* \end{aligned} \tag{2}$$

where $w^*$ is the initial imperfection of the plate, and $\varepsilon_{xx}$, $\varepsilon_{yy}$, and $\gamma_{xy}$ are the normal and shear strains.

Substituting Eq. (1) into the non-linear strain–displacement relations (2) gives the kinematic relations as

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} \tag{3}$$

where

$$\begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} = \begin{Bmatrix} u_{0,x} + \frac{1}{2} w_{0,x}^2 + w_{0,x}^* w_{0,x} \\ v_{0,y} + \frac{1}{2} w_{0,y}^2 + w_{0,y} w_{0,y}^* \\ u_{0,y} + v_{0,x} + w_{0,x} w_{0,y} + w_{0,x}^* w_{0,y} + w_{0,x} w_{0,y}^* \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} = \begin{Bmatrix} -w_{0,xx} \\ -w_{0,yy} \\ -2w_{0,xy} \end{Bmatrix} \tag{4}$$

The constitutive equations for the $k$th ply of laminated composite, taking into account the thermal effects are given by [15]

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix}_k = \begin{pmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{21} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{61} & \overline{Q}_{62} & \overline{Q}_{66} \end{pmatrix}_k \begin{pmatrix} \varepsilon_{xx} - \alpha_x \Theta \\ \varepsilon_{yy} - \alpha_y \Theta \\ \gamma_{xy} - \alpha_{xy} \Theta \end{pmatrix}_k \tag{5}$$

and for the piezoelectric layers [15]

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} Q_{11}^a & Q_{12}^a & 0 \\ Q_{21}^a & Q_{22}^a & 0 \\ 0 & 0 & Q_{66}^a \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} - \alpha_x^a \Theta \\ \varepsilon_{yy} - \alpha_y^a \Theta \\ \gamma_{xy} - \alpha_{xy}^a \Theta \end{pmatrix} - \begin{pmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{32} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{6}$$

where $\sigma_{xx}$, $\sigma_{yy}$, and $\tau_{xy}$ are the normal and shear stresses, $\Theta$ the temperature difference with respect to the reference temperature, $\alpha_x$, $\alpha_y$, $\alpha_{xy}$, $\alpha_x^a$, $\alpha_y^a$ and $\alpha_{xy}^a$ the coefficients of thermal expansion, $\overline{Q}_{ij}, Q_{ij}^a (i,j = 1,2,6)$ the elastic stiffness, $e_{31}$ and $e_{32}$ the piezoelectric stiffness, $E_x$, $E_y$ and $E_z$ the electric field components, and letter "$a$" means "actuator". The coefficients of thermal expansion, elastic stiffness, piezoelectric stiffness and electric field components are provided in detail in Appendix A. Using the constitutive equations, the stress resultants are found to be

$$\begin{aligned} \begin{Bmatrix} N_x & M_x \\ N_y & M_y \\ N_{xy} & M_{xy} \end{Bmatrix} &= \begin{Bmatrix} Q_{11}^a & Q_{12}^a & 0 \\ Q_{12}^a & Q_{22}^a & 0 \\ 0 & 0 & Q_{66}^a \end{Bmatrix} \left( \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} (I_1^*, 0) + \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} (0, I_2^*) \right) \\ &\quad + \left( (A_{ij}^*, 0) \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + (0, D_{ij}^*) \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} \right) \\ &\quad - \begin{Bmatrix} e_{31} \\ e_{32} \\ 0 \end{Bmatrix} (V_a^*, 0) - \begin{Bmatrix} N_x^T & M_x^T \\ N_y^T & M_y^T \\ N_{xy}^T & M_{xy}^T \end{Bmatrix} \quad i,j = 1,2,6 \end{aligned} \tag{7}$$

where the thermal stress resultants and the parameters $I_1{}^*$, $I_2{}^*$, $V_a^*$, $A_{ij}^*$ and $D_{ij}^*$ are given in detail in Appendix B.

The equilibrium equations of the rectangular plate of laminated composite with piezoelectric actuators may be derived on the basis of the stationary potential energy criterion. The total potential

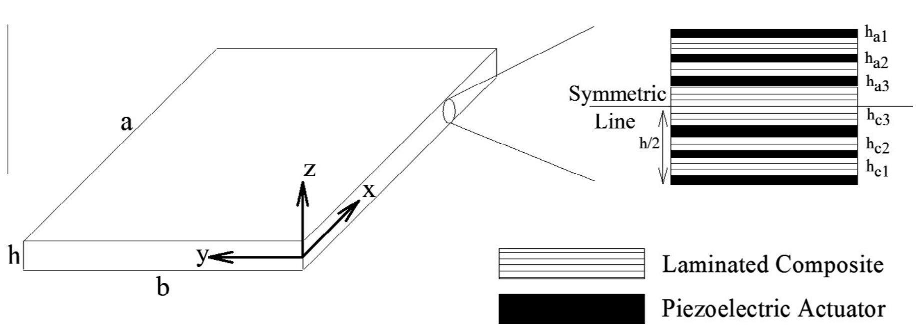


**Fig. 1.** The geometry and structure of the piezolaminated plate.