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Torsion of functionally graded nonlocal viscoelastic circular nanobeams

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ABSTRACT

The elastostatic problem of functionally graded circular nanobeams under torsion, with nonlocal elastic behavior proposed by Eringen, is preliminarily formulated. Exact solutions are detected for nanobeams with arbitrary axial gradations of elastic properties and radially quadratic distributions of shear moduli. Extension of the treatment to nonlocal viscoelastic composite circular nanobeams is then performed. An effective solution procedure based on Laplace transform is developed, providing a new correspondence principle in nonlocal viscoelasticity for functionally graded materials. Displacements, shear strains and stresses are established for nonlocal viscoelastic nanobeams made of periodic fiber-reinforced materials, with polymeric matrix described by a Maxwell model connected in series with a Voigt model.

1. Introduction

Analysis of functionally graded beams under torsion is a research topic of major interest in engineering applications. Nevertheless, exact solutions are available only for special cross-sections and gradations of elastic properties. Computational strategies and effective homogenization techniques [1–18] are thus usually adopted in order to analyze and design such structures. An elegant solution procedure, based on a modified version of Lekhnitskii formalism [19], was proposed in [20] for cylindrically anisotropic beams. Applications on laminates and novel solutions for circular cylindrical bars, also with angular symmetry, were investigated in [21,22]. The effects of material inhomogeneities on the torsional response of linearly elastic isotropic bars were assessed in [23]. Analytical stress solutions for composite cylinders were given in [24]. Further solutions, with special emphasis on end effects, were assessed in [25,26] for cylindrically anisotropic circular tubes and bars under thermal and mechanical loadings. Functionally graded beams with shear moduli, defined by positive functions of the Prandtl stress function of corresponding elastically homogeneous beams were analyzed in [27,28]. Inhomogeneous hollow cylinders made also of isotropic and incompressible linearly elastic materials were studied in [29–31]. However, in these contributions the constitutive behavior is elastically local, with cross-sectional inhomogeneities, see also [32–37]. An exception was dealt with in [38] still for local cylinders, but axially graded. The motivation of the present manuscript is in answering the question: "Is it possible to detect new exact solutions for composite viscoelastic nonlocal nanobeams under torsion?" The conclusion is affirmative for circular nanobeams with radially quadratic distributions of shear moduli.

The plan is the following. Basic notations, assumptions and equilibrium conditions governing circular beams are collected in Section 2. The nonlocal elastic equilibrium problem of functionally graded isotropic nonlocal nanobeams is formulated in Section 3. The closed-form expression of nonlocal stresses is provided in Section 4 by resorting to a Saint-Venant-type semi-inverse approach. These fields are then transformed in Laplace domain in Section 5 to solve viscoelastic nanobeams governed by the Eringen nonlocal law given in Section 6. Analytical solutions are detected in Section 7 for composite nonlocal viscoelastic cylinders made of periodic fiber-reinforced materials, with polymeric matrix represented by a Maxwell model connected in series with a Voigt model.

2. Preliminary assumptions and equilibrium conditions

Let us consider a circular domain Ω of radius R describing the cross-section of a straight cantilever subjected to a torque $\mathcal{M}$ at the free-end. Body forces are assumed to vanish and the beam lateral mantle is considered to be traction-free [39]. We denote by $\mathbf{r}$ the radius vector in the cross-section plane $\boldsymbol{\pi}_\Omega$ originating at the centroid $\mathbf{G}$ and by $\mathbf{k}$ the unit vector of the beam z-axis thru $\mathbf{G}$. $\mathbf{R}$ is the linear operator performing the rotation in $\boldsymbol{\pi}_\Omega$ by $\pi/2$ counterclockwise and V and $\overline{V}$ are the linear spaces of translations

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associated with the Euclid 3-D space $\mathcal{S}$ and with $\boldsymbol{\pi}_\Omega$. Following Clebsh [40], we conjecture that the normal interactions between longitudinal fibers of the beam vanish so that the Cauchy stress tensor $\mathbf{T}(\mathbf{r}, z, t)$ is expressed, in terms of the shear stress vector $\boldsymbol{\tau}(\mathbf{r}, z, t)$ on the cross-section at the point $(\mathbf{r}, z)$, by

$$\mathbf{T}(\mathbf{r}, z, t) = \begin{bmatrix} \mathbf{O} & \boldsymbol{\tau} \\ \boldsymbol{\tau}^T & 0 \end{bmatrix} (\mathbf{r}, z, t), \tag{1}$$

where t stands for time. Due to the the absence of body forces, the Cauchy differential condition of equilibrium gives

$$\boldsymbol{\tau}'(\mathbf{r}, z, t) = \mathbf{o}, \qquad \text{div}\ \boldsymbol{\tau}(\mathbf{r}, z, t) = 0, \tag{2}$$

with the prime $(\bullet)'$ denoting partial derivative along the z-axis and div divergence with respect to the position vector $\mathbf{r}$. Since the tractions on the lateral mantle are assumed to vanish, the Cauchy boundary condition of equilibrium takes the form

$$\boldsymbol{\tau}(\mathbf{r}, z, t) \cdot \mathbf{n}(\mathbf{r}, z, t) = 0, \tag{3}$$

where the dot $\cdot$ stands for inner product and $\mathbf{n}$ is the outward unit normal to the cross-section boundary $\partial\Omega$. Let us denote by $\Delta_2 := \text{div}\nabla$ the Laplace operator with respect to the position vector $\mathbf{r}$. As shown in the next section, an useful implication of Eq. (2) is the vanishing of the divergence of the Laplacian of shear stress field

$$(\text{div}\ \Delta_2\boldsymbol{\tau})(\mathbf{r}, z, t) = 0. \tag{4}$$

Indeed, a cartesian evaluation gives

$$\text{div}\ \boldsymbol{\tau} = \tau_{i/i} = 0 \quad \Rightarrow \quad \text{div}\ \Delta_2\boldsymbol{\tau} = \tau_{i/jji} = \tau_{i/ijj} = 0, \tag{5}$$

where the symbol / stands for partial derivative and $i, j \in \{1, 2\}$.

3. Nonlocal isotropic elasticity

The shear stress $\boldsymbol{\tau}(\mathbf{r}, z, t)$ is assumed to be related to the shear strain vector $\boldsymbol{\gamma}(\mathbf{r}, z, t)$ by the isotropic nonlocal elastic law conceived by Eringen [41]

$$\boldsymbol{\tau}(\mathbf{r}, z, t) - (e_o\ a)^2\ \Delta_2\boldsymbol{\tau}(\mathbf{r}, z, t) = \mu(\mathbf{r}, z)\ \boldsymbol{\gamma}(\mathbf{r}, z, t), \tag{6}$$

where e_o is a material constant, a is the internal length. The magnitude of e_o is determined experimentally or approximated by matching the dispersion curves of plane waves with those of atomic lattice dynamics. For single walled carbon nanotubes [42–44] the length scale parameter $c := e_o\ a$ is assessed to be smaller than 2.0 nm [45]. Nonlocal constitutive relations for functionally graded materials have been discussed in [46].

Let us assign the shear modulus in a separable form in $\mathbf{r}$ and z as

$$\mu(\mathbf{r}, z) = \mu_r(\mathbf{r})\ \mu_a(z). \tag{7}$$

The transversal shear modulus is assumed to be radially inhomogeneous according to the quadratic rule

$$\mu_r(\mathbf{r}) := m\|\mathbf{r}\|^2 + k, \quad \text{with} \quad m, k \in \mathcal{R}, \tag{8}$$

where $\|\mathbf{r}\|$ is the norm of the vector $\mathbf{r}$. The shear strain vector at r.h.s. in Eq. (6) is evaluated by conjecturing that the displacement field of the beam under torsion takes the form

$$\mathbf{u}(\mathbf{r}, z, t) = \theta(z, t)\mathbf{R}\mathbf{r}, \tag{9}$$

where θ is the rotation function, about the z-axis, of cross-sections with respect to the clamp.[1] The kinematically compatible deformation writes thus as

$$\mathbf{D}(\mathbf{r}, z, t) = (\text{sym}\ d\mathbf{u})(\mathbf{r}, z, t) = \begin{bmatrix} \mathbf{O} & \boldsymbol{\gamma} \\ \boldsymbol{\gamma}^T & 0 \end{bmatrix} (\mathbf{r}, z, t), \tag{10}$$

with the shear strain vector given by

$$\boldsymbol{\gamma}(\mathbf{r}, z, t) = \theta'(z, t)\mathbf{R}\mathbf{r}. \tag{11}$$

Substituting Eqs. (7) and (11) in Eq. (6), the differential condition of nonlocal elastic kinematic compatibility is expressed as

$$\boldsymbol{\tau}(\mathbf{r}, z, t) - (e_o\ a)^2\ \Delta_2\boldsymbol{\tau}(\mathbf{r}, z, t) = \mu_r(\mathbf{r})\ \mu_a(z)\ \theta'(z, t)\ \mathbf{R}\mathbf{r}. \tag{12}$$

Taking the z-derivative and the $\mathbf{r}$-divergence we get the equations

$$\begin{cases} \boldsymbol{\tau}'(\mathbf{r}, z, t) - (e_o\ a)^2\ \Delta_2\boldsymbol{\tau}'(\mathbf{r}, z, t) = \mu_r(\mathbf{r})\ (\mu_a(z)\ \theta'(z, t))'\ \mathbf{R}\mathbf{r}, \\ (\text{div}\ \boldsymbol{\tau})(\mathbf{r}, z, t) - (e_o\ a)^2\ (\text{div}\ \Delta_2\boldsymbol{\tau})(\mathbf{r}, z, t) = (\mu_a(z)\ \theta'(z, t))'\text{div}\ (\mu_r(\mathbf{r})\ \ \mathbf{R}\mathbf{r}). \end{cases} \tag{13}$$

Resorting to Eq. (8), we get $\nabla\mu_r(\mathbf{r}) \cdot \mathbf{R}\mathbf{r} = 0$. It follows that, being $\text{div}\ \mathbf{R}\mathbf{r} = 0$, also $\text{div}(\mu_r(\mathbf{r})\mathbf{R}\mathbf{r}) = \nabla\mu_r(\mathbf{r}) \cdot \mathbf{R}\mathbf{r} + \mu_r(\mathbf{r})\text{div}\ \mathbf{R}\mathbf{r} = 0$. Recalling Eqs. $(2)_2$ and (4), we conclude that Eq. $(13)_2$ is identically fulfilled. Moreover, imposing the equilibrium Eq. $(2)_1$ in Eq. $(13)_1$, we get

$$(\mu_a(z)\ \theta'(z, t))' = 0, \tag{14}$$

whence the z-constancy condition follows, so that we may set

$$\mu_a(z)\ \theta'(z, t) = \beta(t), \qquad \text{with} \quad \beta(t) \in \mathcal{R}. \tag{15}$$

Eq. (15) was obtained in [38] for radially homogeneous linearly elastic (local) beams, grading the material only along z. The torsional rotation $\theta(z, t)$ is then evaluated by integrating Eq. (15) and by setting $\theta(0) = 0$

$$\theta(z, t) = \beta(t) \int_0^z \frac{1}{\mu_a(\rho)}\ d\rho. \tag{16}$$

As pointed out in [38], the scalar function $\beta(t)$ is computed by imposing the static equivalence condition around the z-axis

$$\mathcal{M}(t) = \int_\Omega \mathbf{R}\mathbf{r} \cdot \boldsymbol{\tau}(\mathbf{r}, z, t)\ dA. \tag{17}$$

The explicit expression of the function $\beta(t)$ is provided in the next section.

4. Nonlocal elastic shear stresses

The shear stress field, solution of the nonlocal elastostatic problem of functionally graded circular nanobeams under torsion formulated in Section 3, is given by the formula

$$\begin{aligned} \boldsymbol{\tau}(\mathbf{r}, z, t) &= \beta(t)(\mu_r(\mathbf{r}) + 8\ c^2 m)\ \mathbf{R}\mathbf{r} \\ &= \beta(t)(m\|\mathbf{r}\|^2 + k + 8\ c^2 m)\mathbf{R}\mathbf{r}. \end{aligned} \tag{18}$$

Note that the the physical dimensions of the parameters m and k are $[FL^{-4}]$ and $[FL^{-2}]$ respectively. The length scale parameter c and the scalar $\beta(t)$ have respectively the physical dimensions of a length and of the inverse of a length. Proof of Eq. (18) consists of two steps.

(1) Check of equilibrium, described by Eqs. (2) and (3).
(2) Check of nonlocal elastic kinematic compatibility Eq. (12).

Let us preliminary provide a list of noteworthy identities

$$\begin{cases} \text{div}\ (\|\mathbf{r}\|^2\mathbf{R}\mathbf{r}) = 0, \\ \nabla(\|\mathbf{r}\|^2\mathbf{R}\mathbf{r}) = 2\mathbf{R}\mathbf{r} \otimes \mathbf{r} + \|\mathbf{r}\|^2\mathbf{R}, \\ \text{div}\ (2\mathbf{R}\mathbf{r} \otimes \mathbf{r}) = 6\mathbf{R}\mathbf{r}, \\ \text{div}\ (\|\mathbf{r}\|^2\mathbf{R}) = 2\mathbf{R}\mathbf{r}, \\ \Delta_2(\|\mathbf{r}\|^2\mathbf{R}\mathbf{r}) = \text{div}\ \nabla(\|\mathbf{r}\|^2\mathbf{R}\mathbf{r}) = 8\mathbf{R}\mathbf{r}. \end{cases} \tag{19}$$

Due to the z-independence of Eq. (18), the differential condition of equilibrium Eq. $(2)_1$ is trivially verified. Eq. $(2)_2$ follows from Eq. $(19)_1$. Fulfillment of the boundary equilibrium Eq. (3) is a direct consequence of the orthogonality condition $\mathbf{R}\mathbf{r} \cdot \mathbf{n}(\mathbf{r}) = 0$, being $\mathbf{n}(\mathbf{r})$

[1] In Saint-Venant theory the rotation θ is affine in the abscissa z [47,48].