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Point charge potential and weighting field of a pixel or pad in a plane condenser

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ABSTRACT

We derive expressions for the potential of a point charge as well as the weighting potential and weighting field of a rectangular pad for a plane condenser, which are well suited for numerical evaluation. We relate the expressions to solutions employing the method of image charges, which allows discussion of convergence properties and estimation of errors, providing also an illuminating example of a problem with an infinite number of image charges.

1. Introduction

In this report we derive the potential of a point charge and the so-called weighting potential and weighting field of a rectangular pad in a parallel plate geometry. These solutions are needed to calculate the signals in e.g. silicon pixel detectors as well as micropattern detectors with pixel or pad readout. The surface charge density σ induced on the metal planes by the presence of the point charge Q is related to the electric field E on the metal surface by $\sigma = \varepsilon_0 E$. Knowing the potential ϕ of a point charge Q at a (possibly time dependent) position x_0, y_0, z_0, the induced charge and current on a rectangular pad centred at zero is therefore given by

$$Q_{ind} = \int_{-w_x/2}^{w_x/2} \int_{-w_y/2}^{w_y/2} -\varepsilon_0 \vec{\nabla} \phi|_{z=0}\, dx\, dy, \quad I_{ind} = -\frac{dQ_{ind}}{dt} \tag{1}$$

Due to Green's reciprocity theorem [1] the charge and current are also given by

$$Q_{ind} = -\frac{Q}{V_w} \phi_w(\vec{x}_0), \quad I_{ind} = -\frac{Q}{V_w} \vec{E}_w(\vec{x}_0)_w \frac{d\vec{x}_0}{dt} \tag{2}$$

where ϕ_w and $E_w = -\vec{\nabla}\phi_w$ are the potential and electric field in the detector volume, respectively, in case all charges in the detector are removed, the pad is put to potential V_w and the rest stays grounded [2,3]. In the following we derive the expressions for ϕ, ϕ_w and E_w.

2. Potential

Fig. 1a shows a point charge between two grounded metal planes at a distance d. The potential is written as ϕ_1 in the region $0 < z < z_0$ and ϕ_2 in the region $z_0 < z < d$. We have

$$\phi_1(r,z) = \frac{Q}{2\pi\varepsilon_0} \int_0^\infty J_0(kr) \frac{\sin h(kz)\ \sin h(k(d-z_0))}{\sin h(kd)}\, dk \tag{3}$$

$$\phi_1(x,y,z) = \frac{Q}{\pi^2\varepsilon_0} \int_0^\infty \int_0^\infty \frac{\cos(k_x x)\ \cos(k_y y)}{k} \frac{\sin h(kz)\ \sin h(k(d-z_0))}{\sin h(kd)}\, dk_x\, dk_y \tag{4}$$

in cylindrical and cartesian coordinates, respectively [1,4]. $J_0(x)$ is the Bessel function of first kind and in Eq. (4) we have defined $k = \sqrt{k_x^2 + k_y^2}$. For ϕ_2 we just have to swap z and z_0. The integrand has an infinite number of complex poles at $k_n = in\pi/d$ and by finding an appropriate contour in the complex plane the integral of Eq. (3) can be expressed as the sum of the residues which evaluates to [1]

$$\phi(r,z) = \frac{Q}{\pi\varepsilon_0 d} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi z}{d}\right) \sin\left(\frac{n\pi z_0}{d}\right) K_0\left(\frac{n\pi r}{d}\right) \tag{5}$$

where $K_0(x)$ is the modified Bessel function of second kind. Since $K_0(x)$ has a logarithmic singularity at $x=0$ the expression diverges for $r=0$ and has slow convergence close to $r=0$. For numerical evaluation it is therefore easier to focus on the integral in Eqs. (3) and (4) for which very efficient methods are available. For large values of k the integrand of Eq. (3) behaves as $\sqrt{2/(kr\pi)}\ \cos(kr - \pi/4)e^{-k(z_0-z)}$. The exponential behaviour therefore allows to set the upper integration limit for k to a multiple of $1/(z_0 - z)$ for precise numerical

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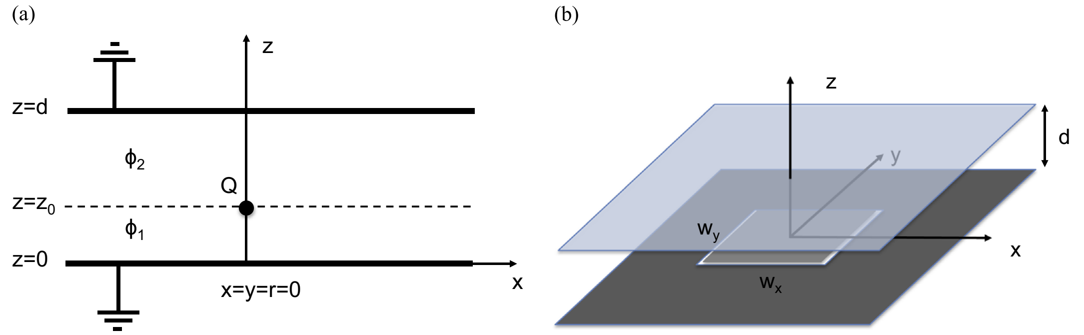

Fig. 1. (a) Point charge Q between two grounded metal planes. (b) Readout pad or pixel of dimension w_x and w_y centred at the origin.

evaluation. For values of $z=z_0$ i.e. in the plane of the point charge, the integral however shows very slow $1/\sqrt{kr}$ decay and numerical evaluation is difficult. We therefore apply the methods discussed in [4] where we subtract one or more exponential terms from the integrand which can be integrated explicitly. We can rewrite part of the integrand in the following form:

$$\begin{aligned}\frac{\sinh(kz)\,\sinh(k(d-z_0))}{\sinh(kd)} &= \frac{1}{2}e^{-k(z_0-z)}-\frac{1}{2}e^{-k(z_0+z)} \\ &+\sum_{n=1}^{N}\left[\frac{1}{2}e^{-k(2nd-z_0+z)}+\frac{1}{2}e^{-k(2nd+z_0-z)}\right. \\ &\left.-\frac{1}{2}e^{-k(2nd-z_0-z)}-\frac{1}{2}e^{-k(2nd+z_0+z)}\right] \\ &-e^{-k(2N+1)d}\,\frac{\sinh(kz)\,\sinh(kz_0)}{\sinh(kd)}\end{aligned} \tag{6}$$

where $N>0$ is an arbitrary positive integer. Inserting this expression into Eqs. (3) and (4) and using the relations [1]

$$\int_0^\infty J_0(kr)e^{-k|z|}\,dk=\frac{1}{\sqrt{r^2+z^2}}$$

$$\frac{2}{\pi}\int_0^\infty\int_0^\infty \cos(k_x x)\,\cos(k_y y)\frac{e^{-k|z|}}{k}\,dk_x\,dk_y=\frac{1}{\sqrt{x^2+y^2+z^2}}$$

we find

$$\begin{aligned}\frac{4\pi\varepsilon_0}{Q}\,\phi(r,z) &= \frac{1}{\sqrt{r^2+(z-z_0)^2}}-\frac{1}{\sqrt{r^2+(z+z_0)^2}} \\ &+\sum_{n=1}^{N}\left[\frac{1}{\sqrt{r^2+(z+2nd-z_0)^2}}+\frac{1}{\sqrt{r^2+(z-2nd-z_0)^2}}\right. \\ &\left.-\frac{1}{\sqrt{r^2+(z-2nd+z_0)^2}}-\frac{1}{\sqrt{r^2+(z+2nd+z_0)^2}}\right] \\ &-\int_0^\infty 2J_0(kr)e^{-k(2N+1)d}\,\frac{\sinh(kz)\,\sinh(kz_0)}{\sinh(kd)}\,dk\end{aligned} \tag{7}$$

and

$$\begin{aligned}\frac{4\pi\varepsilon_0}{Q}\,\phi(x,y,z) &= \frac{1}{\sqrt{x^2+y^2+(z-z_0)^2}}-\frac{1}{\sqrt{x^2+y^2+(z+z_0)^2}} \\ &+\sum_{n=1}^{N}\left[\frac{1}{\sqrt{x^2+y^2+(z+2nd-z_0)^2}}\right. \\ &+\frac{1}{\sqrt{x^2+y^2+(z-2nd-z_0)^2}}-\frac{1}{\sqrt{x^2+y^2+(z-2nd+z_0)^2}} \\ &\left.-\frac{1}{\sqrt{x^2+y^2+(z+2nd+z_0)^2}}\right] \\ &-\int_0^\infty\int_0^\infty \frac{4}{\pi}\,\frac{\cos(k_x x)\,\cos(k_y y)}{k}\,e^{-k(2N+1)d} \\ &\times\frac{\sinh(kz)\,\sinh(kz_0)}{\sinh(kd)}\,dk_x\,dk_y\end{aligned} \tag{8}$$

Since the expressions are symmetric with respect to z and z_0 we do not have to distinguish between ϕ_1 and ϕ_2 and ϕ is therefore valid in the entire range of $0<z<d$. The above expressions represent the potential created by a point charge and $4N+1$ mirror charges together with a remaining integral part: charges of $-Q$ at positions $-z_0$ and $-z_0\pm 2nd$ and charges of $+Q$ at positions z_0 and $z_0\pm 2nd$.

For the maximum possible values of $z, z_0=d$ the remaining integrand behaves as e^{-2kNd}, so for numerical evaluation of the integral an upper integration limit as a multiple of $1/(2Nd)$ will be sufficient for precise evaluation. Since $J_0(kr)\le 1$ the integral part of Eqs. (7) and (8) is always smaller than

$$\int_0^\infty 2e^{-k(2N+1)d}\,\sinh(kd)\,dk=\frac{1}{2d}\frac{1}{N^2+N} \tag{9}$$

so we find the upper limit on the error $\Delta\phi$ of the calculated potential ϕ by terminating the series at N and neglecting the integral to be

$$|\Delta\phi|<Q/(8\pi\varepsilon_0 N^2 d) \tag{10}$$

By bringing N to ∞ the error becomes zero and the field is represented as an infinite number of mirror charges. This also provides the mathematical proof that the procedure of an infinite number of mirror charges converges to the correct potential. By moving the grounded plate at $z=d$ to infinity, only the first two terms in Eqs. (7) and (8) remain, which represents the correct result for a point charge in the presence of a single grounded plane i.e. a charge Q at $z=z_0$ and a single mirror charge of value $-Q$ at $z=-z_0$.

3. Induced charge and weighting field

Using Eqs. (1) and (4) we can now calculate the charge induced on the rectangular pad, as shown in Fig. 1b, according to

$$Q_{ind}(x_0,y_0,z_0)=\int_{-w_x/2}^{w_x/2}\int_{-w_y/2}^{w_y/2}-\varepsilon_0\frac{\partial\phi(x-x_0,y-y_0,z)}{\partial z}\Big|_{z=0}\,dx\,dy \tag{11}$$

With Eq. (2) we can express the result through the weighting potential ϕ_w as

$$\begin{aligned}\phi_w(x,y,z) &= \frac{4V_w}{\pi^2}\int_0^\infty\int_0^\infty \cos(k_x x)\,\sin\left(k_x\frac{w_x}{2}\right)\,\cos(k_y y) \\ &\times\sin\left(k_y\frac{w_y}{2}\right)\frac{\sinh(k(d-z))}{k_x k_y\,\sinh(kd)}\,dk_x\,dk_y\end{aligned} \tag{12}$$

We can now verify that this is indeed equal to the solution of the Laplace equation with boundary condition $\phi_w(x,y,z=0)=V_w$ in