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Systematic optimization of exterior measurement locations for the determination of interior magnetic field vector components in inaccessible regions

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ABSTRACT

An experiment may face the challenge of real-time determination of the magnetic field vector components present within some interior region of the experimental apparatus over which it is impossible to directly measure the field components during the operation of the experiment. As a solution to this problem, we propose a general concept which provides for a unique determination of the field components within such an interior region solely from exterior measurements at fixed discrete locations. The method is general and does not require the field to possess any type of symmetry. We describe our systematic approach for optimizing the locations of these exterior measurements which maximizes their sensitivity to successive terms in a multipole expansion of the field.

1. Introduction

An experiment may face the challenge of determining in real-time the vector components, B_i, of the magnetic field within some region of space over which it is impossible or impractical to carry out a direct measurement of these field components during the operation of the experiment. For example, suppose an experiment requires a highly uniform magnetic field. A suitable magnetic field coil would need to be designed and fabricated and the coil's field would almost certainly be measured directly with a field probe as a verification of the coil's design principles. Suppose that after integration with the experiment the configuration of the apparatus and the coil precludes any such direct measurement (e.g., it may not be physically possible to measure the field components at multiple positions due to geometry constraints). If the nature of the experiment is such that it would be highly desirable to monitor the field components during the operation of the experiment, one then faces such a challenge.

The general idea of our concept, as illustrated schematically in Fig. 1, is to perform measurements of the field components at discrete points on some surface exterior to the experiment's sensitive volume (hereafter, we will refer to this interior sensitive volume as the "fiducial volume"). If the volume interior to this surface encloses no currents or sources of magnetization, such that the current density $\vec{J}=0$ and the magnetization $\vec{M}=0$, it then follows that the vector components of the field, B_i, and the magnetic scalar potential, Φ_M, will each satisfy a Laplace equation, which can then be solved either numerically or analytically, providing for a unique determination of the vector components within the interior fiducial volume. It should be noted that the Laplace equation for each of the vector components of the field, $\vec{\nabla}^2 B_i=0$, holds if the field components are expressed in terms of (B_x, B_y, B_z) rectangular coordinates, but does not hold if the components are expressed in terms of curvilinear coordinates. (For example, in spherical components, (B_r, B_θ, B_ϕ) do not separately obey Laplace equations.)

One approach to the solution of this problem is to cast the problem as a boundary-value problem, under which the vector components are measured over a (regular) grid on the surface [1]. The interior field components can then be determined uniquely via standard numerical techniques (e.g., [2]) for the solution of the Laplace equation. This approach has its limitations; in particular, the concept of a boundary-value problem requires measurements of the boundary values over a grid spanning (nearly) the entire surface, and then the number of interior points at which the field can be determined is limited by the grid spacing (as is the accuracy).

The approach we propose here starts from the general result that the solution to the Laplace equation in spherical coordinates can be written as an expansion in radial coordinates and spherical harmonics. If the values of the field components or the scalar potential (equivalently, the component of the field normal to the surface) are known at some number of exterior fixed points, a system of linear equations can be constructed and subsequently solved for the *a priori* unknown expansion coefficients in the multipole expansion. This then provides for a unique

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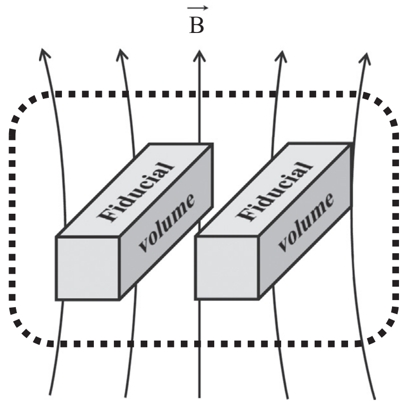

Fig. 1. Schematic illustration of the general idea of our concept. The vector field components in the inaccessible interior region of the experiment (the fiducial volumes) are determined from measurements of the field components on some surface (indicated by the dashed line) exterior to the fiducial volume.

determination of the field components in the interior region. Of course, the above is well-known (e.g., [3]); however, what is novel about our approach is that we have developed a systematic method for optimizing the positions at which the field components are to be measured which maximizes their sensitivity to the successive terms in the multipole expansion, and which also permits for discrimination between these successive terms. To our knowledge, our method has not been published elsewhere, although there has been previous work which proposed a less general approach whereby measurements of boundary values over the six faces of a rectangular volume were fitted to trigonometric and hyperbolic functions, thus determining the interior field components [4].

The remainder of this paper is organized as follows. We begin, in Section 2 with a discussion of the mathematical details of our method. Then, in Section 3, we show example results from applications of our method to various magnetic field profiles. Finally, we conclude with a brief summary in Section 4.

2. Method

As is well known (e.g., [3]), the general solution to the Laplace equation, $\vec{\nabla}^2 f(\vec{x}) = 0$, where for our problem we have $f \in \{B_x, B_y, B_z, \Phi_M\}$, can be written in spherical coordinates as a multipole expansion. In Appendix A, we reconcile the solution to the Laplace equation in which the m values range over $m = -\ell, -\ell+1, \ldots, +\ell$ with the often-quoted and utilized form (e.g., [5]) in which the m values instead range over only non-negative integers $m = 0, 1, \ldots, +\ell$. Such a solution is of the form

$$f(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{+\ell} r^\ell P_\ell^m(\cos\theta)[a_{\ell m}\cos m\phi + b_{\ell m}\sin m\phi], \qquad (1)$$

where the P_ℓ^m are the associated Legendre polynomials, and the $a_{\ell m}$ and $b_{\ell m}$ are arbitrary expansion coefficients which reflect the intrinsic properties of the field. For $f = B_x$, B_y, or B_z, we rewrite Eq. (1) as

$$B_i(\vec{x}) = \sum_{\ell,m} C_{\ell m}(\vec{x}) a_{\ell m,\,i} + S_{\ell m}(\vec{x}) b_{\ell m,\,i}, \qquad (2)$$

where we define the $C_{\ell m}$ and $S_{\ell m}$ to be "basis functions" of the form

$$C_{\ell m}(\vec{x}) = r^\ell P_\ell^m(\cos\theta)\cos(m\phi),$$
$$S_{\ell m}(\vec{x}) = r^\ell P_\ell^m(\cos\theta)\sin(m\phi). \qquad (3)$$

2.1. Field component method

Suppose one can measure $B_i(\vec{x})$ at N different locations in a region exterior to the fiducial volume. We use $B_i(\vec{x}_n)$ to denote an exterior measurement of B_i at location $\vec{x}_n$, where the index $n = 1, \ldots, N$. It then follows that we can construct a $N \times N$ system of equations for the $a_{\ell m,i}$ and $b_{\ell m,i}$ expansion coefficients,

$$B_i(\vec{x}_n) = \sum_{\ell,m=0}^{L,M} C_{\ell m}(\vec{x}_n) a_{\ell m,\,i} + S_{\ell m}(\vec{x}_n) b_{\ell m,\,i}, \quad n = 1, \ldots, N, \qquad (4)$$

as the $B_i(\vec{x}_n)$ are known (i.e., measured) quantities, and the $C_{\ell m}(\vec{x}_n)$ and $S_{\ell m}(\vec{x}_n)$ basis functions are known functions of $\vec{x}_n$. The upper limits L and M denote the maximum values of ℓ and m permitted for N unknowns. For example, if $N=5$, the possible (ℓ, m) values would be: (0, 0), (1, 0), (1, 1), (2, 0), (2, 1); that is, the series would be truncated at $(L, M) = (2, 1)$. We note that our convention is simply to employ the N lowest-order (ℓ, m) combinations. For some particular field profile it might be the case that the selection of some non-incremental combination would be advantageous, such as (2, 2) instead of (2, 0) as in the $N=5$ example above. But, then for some different field profile, the selection of that non-incremental term might then not be optimal.

The resulting $N \times N$ system of linear equations can then be readily solved via standard numerical methods (e.g., Gaussian elimination). The field component B_i everywhere within the interior fiducial volume can then be calculated from the $a_{\ell m,i}$ and $b_{\ell m,i}$ expansion coefficients. Note that one drawback of this method, which we hereafter refer to as our "field component method", is that exterior measurements of B_i then provide only limited information on the interior values of $B_{j\neq i}$. For example, suppose that exterior measurements are made of the B_x component. The method would then determine $B_x(\vec{x})$, and thus the $\partial B_x/\partial x$, $\partial B_x/\partial y$, and $\partial B_x/\partial z$ gradients. The constraint $\vec{\nabla} \times \vec{B} = 0$ would then permit determination of two more gradients, $\partial B_z/\partial x$ and $\partial B_y/\partial x$. Another drawback of this method is illustrated with the following example. Suppose one could carry out the exterior measurements at $N=5$ different locations with a field probe which provides information on all three components of the field, (B_x, B_y, B_z). Although one could then reconstruct all three components within the interior fiducial volume, the expansion for each field component would be limited to $(L, M) = (2, 1)$; that is, the $5 \times 3 = 15$ different field component measurements would not permit for a determination of (ℓ, m) terms of higher order than (2, 1).

2.2. Scalar potential method

Alternatively, if we take $f(\vec{x}) = \Phi_M(\vec{x})$ in the Laplace equation $\vec{\nabla}^2 f(\vec{x}) = 0$, the general solution for the magnetic scalar potential $\Phi_M(\vec{x})$ can, of course, also be written as a multipole expansion like Eq. (1). Using $\vec{B} = -\vec{\nabla}\Phi_M$, it follows that the magnetic field in (B_r, B_θ, B_ϕ) spherical components is of the form

$$B_r(\vec{x}) = -\frac{1}{r}\sum_{\ell,m}[\ell C_{\ell m}(\vec{x}) a_{\ell m} + \ell S_{\ell m}(\vec{x}) b_{\ell m}],$$
$$B_\theta(\vec{x}) = -\frac{1}{\sin\theta}\sum_{\ell,m}[\Delta_{\ell m}(\vec{x}) a_{\ell m} + \Lambda_{\ell m}(\vec{x}) b_{\ell m}],$$
$$B_\phi(\vec{x}) = -\frac{1}{r\sin\theta}\sum_{\ell,m}[m C_{\ell m}(\vec{x}) b_{\ell m} - m S_{\ell m}(\vec{x}) a_{\ell m}], \qquad (5)$$

where the $C_{\ell m}(\vec{x})$ and $S_{\ell m}(\vec{x})$ basis functions are as defined previously in Eq. (3), and the additional basis functions $\Delta_{\ell m}(\vec{x})$ and $\Lambda_{\ell m}(\vec{x})$ are defined in terms of $C_{\ell m}(\vec{x})$ and $S_{\ell m}(\vec{x})$ as

$$\Delta_{\ell m}(\vec{x}) = \frac{\ell}{r} C_{\ell m}(\vec{x})\cos\theta - (\ell+m) C_{\ell-1,m}(\vec{x}),$$
$$\Lambda_{\ell m}(\vec{x}) = \frac{\ell}{r} S_{\ell m}(\vec{x})\cos\theta - (\ell+m) S_{\ell-1,m}(\vec{x}). \qquad (6)$$