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# A remarkable focusing property of a parabolic mirror for neutrons in the gravitational field: Geometric proof

S. Masalovich *

*Technische Universität München, Heinz Maier-Leibnitz Zentrum (MLZ, FRM II), Lichtenbergstr. 1, D-85747 Garching, Germany*

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ABSTRACT

An extraordinary focusing property of a parabolic mirror for ultracold neutrons in the presence of the gravitational field was first reported by Steyerl and co-authors. It was shown that all neutrons emitted from the focus of the mirror will be reflected back upon the same focus passing a point of return in the gravitational field in between. The present note offers a complementary geometric proof of this feature and discusses its application to many-mirror systems. The results can also be applied to electrons and ions in an electric field.



## 1. Introduction

In 1985, the group of scientists led by Steyerl at the Technische Universität München reported successful operation of a unique neutron microscope using ultracold neutrons (UCN) [1]. The ability of such neutrons to reflect specularly from surfaces of many materials at any angle of incidence makes it possible, in principle, to use common optical schemes for designing a neutron reflecting microscope. However, unlike common optics, in UCN optics neutron trajectories are curved by the Earth's gravity. The curvature of every trajectory (flight parabola) depends on the neutron velocity and the initial angle at an object thus resulting in appearance of considerable chromatic aberrations.

With the aim to correct these aberrations the authors developed a very special two-mirror optical system with a common vertical axis. In this system neutrons first pass through the object and move downwards, then reflect upwards from the concave parabolic mirror, pass the highest points of flight parabolas and, on their way down, reflect from the convex spherical mirror. The achromatism of that system is mainly resulted from the unique property of the focal point of a parabolic mirror discovered by the authors. In the first paper [1] this property was reported for paraxial rays, but in the next paper [2] it was already stated: "A particle emanating from the focal point in any direction and at any given speed is reflected back into its origin by the mirror which we assume to be infinite extent". In addition, the authors reported "the unique property of constant flight-time between the coincident object and image, irrespective of initial flight direction". Both properties were confirmed by extensive algebraic calculations and computer simulations.

In this note we present a geometric proof of the extraordinary property of a parabolic mirror mentioned above. Although the proof below will be given for ultracold neutrons in the Earth's gravitational field, it remains true for any massive particles in a homogeneous force field (e.g., an electron or ion in an external electric field).

## 2. Geometric proof

First we prove that the focus of a parabolic mirror will be imaged back upon itself by that mirror. Because of the axial symmetry we only consider trajectories in the plane through the axis of the parabolic mirror. The equation for the mirror's surface in the frame of reference, in which the origin is located at the focus, is given by

$$z = \frac{x^2}{4f} - f, \tag{1}$$

where $f$ is the focal length of the parabolic mirror. The parabola opens up and the optical axis is arranged along the vertical axis $z$, whereas the axis $x$ is arranged horizontally. For a neutron emitted

* Tel.: +49 89 289 14559; fax: +49 89 289 14997.
*E-mail address:* Sergey.Masalovich@frm2.tum.de

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from the origin with the velocity $v$ and angle α with the vertical axis the equation of the trajectory can be written in a parametric form

$$x(t) = v \cdot \sin \alpha \cdot t$$
$$z(t) = v \cdot \cos \alpha \cdot t - \frac{gt^2}{2}. \quad (2)$$

Here $g$ is the acceleration due to gravity and $t$ is a flight time. By eliminating $t$, we modify Eq. (2) into

$$z_1 = \frac{2g}{4v^2 \sin^2\alpha} \cdot x_1^2 - \frac{v^2 \sin^2\alpha}{2g} \quad (3)$$

where

$$x_1 = x - \frac{v^2 \sin 2\alpha}{2g}$$
$$z_1 = -z + \frac{v^2 \cos 2\alpha}{2g}. \quad (4)$$

A comparison of Eqs. (1) and (3) shows that the neutron trajectory is a parabola with the focal length equal to $v^2 \sin^2\alpha/2g$. From Eq. (4) we also find that the coordinates of the focus of that flight parabola are: $(v^2 \sin 2\alpha/2g, v^2 \cos 2\alpha/2g)$. Thus we arrive at a conclusion (well-known, for example, in electron optics [3] for electrons in a homogeneous electric field) that focuses of all parabolic trajectories of particles emitted from the origin with the velocity $v$ lie on the circle of radius $R = v^2/2g$ with the center at the origin.

The imaging properties of a parabolic mirror for neutrons in the Earth's gravitational field can now be examined. As an example, Fig. 1 shows the neutron trajectory (curve I) running through the focus O of the parabolic mirror and the trajectory of the same neutron (curve II) after being reflected specularly from the mirror at point A.

Let us denote by $v_0$ the neutron velocity at point O. Then, as it follows from the above results, the focus B of the parabola I lies on the circle of radius $R_0 = v_0^2/2g$ with the center at O. Since the parabolas I and II pass through the common point A, and at this point they both are characterized by the velocity $v_A$, then the focuses of these two parabolas lie on the same circle of radius $R_A = v_A^2/2g$ with the center at A. The focus of the curve II is marked by letter C as shown in Fig. 1. We are to show that C lies on the circle $R_0$. Let us draw the vertical line DE through the point A, which is parallel to the axes of the parabolas I and II, and parabolic mirror. Besides, through this point we draw tangents to the curve I (AF), to the curve II (AG), and to the parabolic mirror (HK).

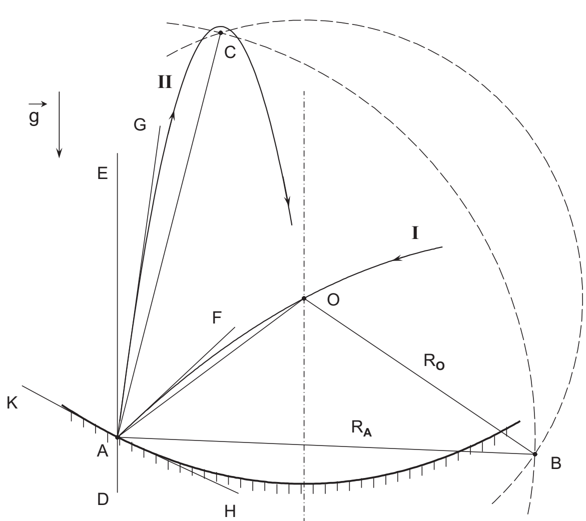

**Fig. 1.** Reflection of a neutron by a parabolic mirror.

Due to specular reflection one can write

$$\angle KAG = \angle HAF. \quad (5)$$

By the properties of a parabola, we also have

$$\angle KAE = \angle DAH = \angle HAO, \quad (6)$$

$$\angle BAF = \angle FAE, \quad (7)$$

$$\angle GAE = \angle GAC. \quad (8)$$

From Eqs. (5) and (6) it follows that

$$\angle GAE = \angle FAO. \quad (9)$$

Rewriting Eq. (7) in terms of the results of Eqs. (8) and (9), we obtain

$$\angle CAO = \angle BAO. \quad (10)$$

Hence we have shown that points B and C not only lie on the same circle with the center at A, but also they are symmetric about the line OA. Since the point B belongs to the circle with the center at O, then the point C belongs to the same circle and $OC = v_0^2/2g$ as shown in Fig. 1. We now prove that the curve II passes through O. Indeed, by the energy conservation law, the neutron velocity equals $v_0$ at the point of intersection of the trajectory with the horizontal line passing through O. Assuming now that the neutron trajectory intersects this horizontal line not at O but at some neighboring point $O_1$, we arrive to the conclusion that $O_1C$ has to be equal to $v_0^2/2g$. Since we have already shown that $OC = v_0^2/2g$, then it follows that $O_1$ coincides with O and the reflected parabola II passes through the focus of the mirror. Recalling that as initial conditions we took an arbitrary neutron velocity and angle with the vertical axis at O, we conclude that the proved statement is true for any neutron trajectory.

It seems to be practically important to extend our consideration to include the multiple successive reflections of a neutron bouncing on the mirror. The simple inspection of the problem in this case reveals the fact that the parabolas, which represent segments of the neutron trajectory between successive reflections from the mirror, converge to the vertical line through the focus as the number of reflections increases. This also remains true for the neutron trajectories that initially moved away from the optical axis (see, for example, the time reversed trajectory in Fig. 1). If the parabolic mirror has a small hole at the vertex then the neutron will finally leave the mirror through this hole making a very small angle with the optical axis. In this sense the property of the focal point of a parabolic mirror for neutrons in the gravitational field resembles the one for light optics: the rays, emitted from the focus in all directions, after reflections will be directed parallel to the axis of the mirror. However, unlike light optics, the beam of massive particles in a homogeneous field will finally represent not a wide parallel beam but a narrow slightly divergent beam defined by the size of the hole.

Yet another important conclusion can be derived from the above results. If a neutron does not pass through the focus of a parabolic mirror at the beginning of its movement, then it will never pass through this point in the course of multiple successive reflections from the mirror. This agrees well with the results published by Wallis et al. [4] who studied the atomic cavity based on a parabolic mirror in the gravitational field. In that paper both quantum and classical analysis of the atom bouncing on the parabolic mirror were presented without giving a special consideration to the trajectories through the focus. For other trajectories it was shown that they are confined between the surface of the parabolic mirror and two other parabolic surfaces (envelopes) with all three focuses lying at the same point. As a result, two different situations may arise: Either both parabolas (envelopes) open down, or one parabola opens down while other opens up.