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Explicit solutions for the modeling of laminated composite plates with arbitrary stacking sequences

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ABSTRACT

In this paper, a method to compute explicit solutions for laminated plate with arbitrary stacking sequences is presented. This technique is based on the construction of an a posteriori Reduced-Order Model using the so-called Proper Generalized Decomposition. The displacement field is approximated as a sum of separated functions of the in-plane coordinates x, y, the transverse coordinate z and the orientation of each ply θ_i. This choice yields to an iterative process that consists of solving a 2D and some 1D problems successively at each iteration. In the thickness direction, a fourth-order expansion in each layer is considered. For the in-plane description, classical Finite Element method is used. The functions of θ_i are discretized with linear interpolations. Mechanical tests with different numbers of layers are performed to show the accuracy of the method.

1. Introduction

Composite and sandwich structures are widely used in the industrial field due to their excellent mechanical properties, especially their high specific stiffness and strength. In this context, they can be subjected to severe mechanical loads. For laminated composite design, accurate knowledge of displacements and stresses is required. Moreover, the choice of the stacking sequences has an important influence on the behavior of the structures. The classical way consists in performing different computations with a fixed value of each orientation of the plies. The present approach based on the Proper Generalized Decomposition (PGD) aims at building the explicit solutions with respect to any stacking sequences avoiding the computational cost of numerous computations.

According to published research, various theories in mechanics for the modeling of composite structures have been developed. On the one hand, the Equivalent Single Layer approach (ESL) in which the number of unknowns is independent of the number of layers, is used. But, the transverse shear and normal stresses continuity on the interfaces between layers are often violated. We can distinguish the classical laminate theory [1] (unsuitable for composites and moderately thick plates), the first order shear deformation theory [2], and higher order theories with displacement [3–10] and mixed [11,12] approaches. On the other hand, the Layerwise approach (LW) aims at overcoming the restriction of the ESL concerning the discontinuity of out-of-plane stresses on the interface layers and taking into account the specificity of layered structure. But, the number of degrees of freedom (dofs) depends on the number of layers. We can mention the folllowing contributions [13–17] within a displacement based approach and [18,11,19] within a mixed formulation. As an alternative, refined models have been developed in order to improve the accuracy of ESL models avoiding the additional computational cost of LW approach. So, a family of models, denoted zig-zag models, was derived in [20–22] from the studies described in [23,24]. Note also the refined approach based on the Sinus model [25–27]. This above literature deals with only some aspects of the broad research activity about models for layered structures and corresponding finite element formulations. An extensive assessment of different approaches has been made in [28–32].

Over the past years, the so-called Proper Generalized Decomposition (PGD) [33] has shown interesting features in the reduction model framework. A separated representation of variables called radial approximation was also introduced in the context of the LATIN method [34] for reducing computational costs and used to solve parametric problems [35]. For the scope of our study, the PGD has been successfully applied for the modeling of composite beams and plates [36–38]. The principle of the method consists in using separated representation of the unknown fields to build an approximate solution of the partial differential equations. A first attempt to obtain analytical solutions for composite plate based on a Navier-type solution with a separation of variables can also be found in [39].

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This work aims at modeling composite plate structures regardless of the stacking sequences. For this purpose, the present approach is based on the separation representation where the displacements are written under the form of a sum of products of (i) bidimensional polynomials of (x, y), (ii) unidimensional polynomials of z and (iii) unidimensional polynomials of the orientations of the plies θ_i. As in [38], a piecewise fourth-order Lagrange polynomial of z is chosen as it is particularly suitable to model composite structures. As far as the variation with respect to the in-plane coordinates is concerned, a 2D eight-node quadrilateral Finite Element (FE) is employed. The functions of the orientations of each ply are piecewise linear. Finally, the deduced non-linear problem implies the resolution of $NC+2$ linear problems alternatively (NC is the number of layers). This process yields to a 2D and 1D problems in which the number of unknowns is smaller than in a Layerwise approach. Finally, the solution depends explicitly on the fibers orientation of each ply.

We now outline the remainder of this article. First, the reference mechanical formulation is recalled. Then, the resolution of the parametrized problem by the PGD is described. The particular assumption on the displacements yields a non-linear problem which is solved by an iterative process. Then, the FE approximations are described. Finally, numerical tests are performed. A one-layer case is first considered to assess and illustrate the behavior of the method. The influence of the discretization of the functions depending on the orientation of the plies is shown. Two-layer and four-layer configurations are also addressed. The accuracy of the results is evaluated by comparison with reference solutions issued from PGD solutions with different fixed stacking sequences using the work developed in [38].

2. Reference problem description

2.1. The governing equations

Let us consider a plate occupying the domain $\mathcal{V}=\Omega\times\Omega_z$ bounded by $\partial\Omega$ with $\Omega_z=\left[-\frac{h}{2},\frac{h}{2}\right]$ in a Cartesian coordinate system (x,y,z). The plate is defined by an arbitrary region Ω, in the (x,y) plane, located at the midplane $z=0$, and by a constant thickness h. See Fig. 1.

2.1.1. Constitutive relation

The plate can be made of NC perfectly bonded orthotropic layers of the same material. Using matrix notation, the three dimensional constitutive law in the material coordinate is given by:

$$\boldsymbol{\sigma}_m=\widetilde{\mathbf{C}}\boldsymbol{\varepsilon}_m \tag{1}$$

where the stress vector $\boldsymbol{\sigma}_m$, the strain vector $\boldsymbol{\varepsilon}_m$ and $\widetilde{\mathbf{C}}$ are written in the material coordinate. $\widetilde{\mathbf{C}}$ is defined as:

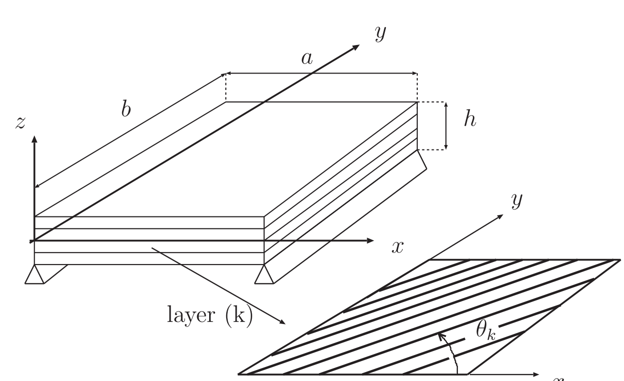

Fig. 1. Geometry of the plate and orientation of the fibers of the laminated plate.

$$\widetilde{\mathbf{C}}=\begin{bmatrix}\widetilde{C}_{11} & \widetilde{C}_{12} & \widetilde{C}_{13} & 0 & 0 & 0\\ & \widetilde{C}_{22} & \widetilde{C}_{23} & 0 & 0 & 0\\ & & \widetilde{C}_{33} & 0 & 0 & 0\\ & & & \widetilde{C}_{44} & 0 & 0\\ & sym & & & \widetilde{C}_{55} & 0\\ & & & & & \widetilde{C}_{66}\end{bmatrix} \tag{2}$$

The stiffness coefficients $\widetilde{C}_{ij}$ are not reported for brevity reason (see [40] for more details). For each layer (k), a rotation matrix $\mathbf{R}^{(k)}$ is defined to obtain the constitutive law in the global coordinate system. So, we have:

$$\begin{bmatrix}\sigma_{11}^{(k)}\\ \sigma_{22}^{(k)}\\ \sigma_{33}^{(k)}\\ \sigma_{23}^{(k)}\\ \sigma_{13}^{(k)}\\ \sigma_{12}^{(k)}\end{bmatrix}=\begin{bmatrix}C_{11}^{(k)} & C_{12}^{(k)} & C_{13}^{(k)} & 0 & 0 & C_{16}^{(k)}\\ & C_{22}^{(k)} & C_{23}^{(k)} & 0 & 0 & C_{26}^{(k)}\\ & & C_{33}^{(k)} & 0 & 0 & C_{36}^{(k)}\\ & & & C_{44}^{(k)} & C_{45}^{(k)} & 0\\ & sym & & & C_{55}^{(k)} & 0\\ & & & & & C_{66}^{(k)}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}^{(k)}\\ \varepsilon_{22}^{(k)}\\ \varepsilon_{33}^{(k)}\\ \gamma_{23}^{(k)}\\ \gamma_{13}^{(k)}\\ \gamma_{12}^{(k)}\end{bmatrix}\ \text{i.e. } \boldsymbol{\sigma}^{(k)}=\mathbf{C}^{(k)}\boldsymbol{\varepsilon}^{(k)} \tag{3}$$

with $\mathbf{C}^{(k)}=\mathbf{R}^{(k)}\widetilde{\mathbf{C}}\mathbf{R}^{(k)^T}$

and

$$\mathbf{R}^{(k)}=\begin{bmatrix}\cos^2\theta_k & \sin^2\theta_k & 0 & 0 & 0 & -2\sin\theta_k\cos\theta_k\\ \sin^2\theta_k & \cos^2\theta_k & 0 & 0 & 0 & 2\sin\theta_k\cos\theta_k\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \cos\theta_k & \sin\theta_k & 0\\ 0 & 0 & 0 & -\sin\theta_k & \cos\theta_k & 0\\ \sin\theta_k\cos\theta_k & -\sin\theta_k\cos\theta_k & 0 & 0 & 0 & \cos^2\theta_k-\sin^2\theta_k\end{bmatrix} \tag{4}$$

We denote the stress vector $\boldsymbol{\sigma}^{(k)}$, the strain vector $\boldsymbol{\varepsilon}^{(k)}$, the fibers orientation θ_k (Fig. 1) and $C_{ij}^{(k)}$ the three-dimensional stiffness coefficients of the layer (k) in the global coordinate system.

2.1.2. The classical weak form of the boundary value problem

The plate is submitted to a surface force density $\mathbf{t}$ defined over a subset Γ_N of the boundary and a body force density $\mathbf{b}$ defined in Ω. We assume that a prescribed displacement $\mathbf{u}=\mathbf{u}_d$ is imposed on $\Gamma_D=\partial\Omega-\Gamma_N$.

The classical formulation of the elastic problem is recalled: find a displacement field $\mathbf{u}$ and a stress field $\boldsymbol{\sigma}$ defined in $\mathcal{V}$ which verify:

- the kinematic constraints:

$$\mathbf{u}\in\mathcal{U} \tag{5}$$

- the equilibrium equations:

$$\boldsymbol{\sigma}\in\mathcal{S}\text{ and }\forall\mathbf{u}^*\in\mathcal{U}^*\\ -\int_{\mathcal{V}}\boldsymbol{\sigma}:\boldsymbol{\varepsilon}(\mathbf{u}^*)\,d\mathcal{V}+\int_{\mathcal{V}}\mathbf{b}.\mathbf{u}^*\,d\mathcal{V}+\int_{\Gamma_N}\mathbf{t}.\mathbf{u}^*d\Gamma=0 \tag{6}$$

- the constitutive relation:

$$\boldsymbol{\sigma}=\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u}) \tag{7}$$

$\mathcal{U}=\{\mathbf{u}\,|\,\mathbf{u}\in(\mathcal{H}^1(\mathcal{V}))^3;\mathbf{u}=\mathbf{u}_d\text{ on }\Gamma_D\}$ is the space in which the displacement field is being sought, $\mathcal{S}=\mathcal{L}^2[\mathcal{V}]^3$ the space of the stresses, and $\boldsymbol{\varepsilon}(\mathbf{u})$ denotes the linearized strain associated with the displacement.

3. Application of the Proper Generalized Decomposition to plate with any stacking sequences

The Proper Generalized Decomposition (PGD) was introduced in [33] and is based on an *a priori* construction of separated variables