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Three-layered plate: Elasticity solution

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ABSTRACT

The subject of this paper is the plate composed of two identical isotropic outer layers and a more compliant inner interlayer, perfectly connected to one another at the interface (three-layered plate). This paper presents a model that describes the behavior of this plate by a system of exact analytical (explicit) equations.

An analytical model is preferred over finite element models and simplified formulas if it is fast and easy-to-use. Thus, modeling has been developed within the framework of two-dimensional elasticity, instead of three. In so doing, the model also represents a means for attaining full comprehension of the involved phenomena, something that neither three-dimensional elasticity nor finite element models and simplified formulas can attain. The two-dimensional behavior is governed here by using assumptions that do not impose constraints on the behavior. Starting from these assumptions, the paper illustrates the relationships between displacements and interface stresses. The subsequent sections of the paper describe the model and present some real case applications.

The contribution of this paper is to consider both the shear modulus and the elastic modulus of the interlayer. Thus, this model applies to three-layered plates with any interlayer, whether utterly compliant or relatively stiff. Conversely, the previous exact analytical models assumed zero elastic modulus, and hence they applied to utterly compliant interlayers only. Hence, not only does the new model predict the exact behavior of plates that the former analytical models described only approximately, but this model may also be used as a benchmark for finite element models, which cannot assign zero value to the elasticity modulus of the interlayer together with the actual shear modulus.

1. Introduction

This paper deals with the analytical modeling of the plate made up of two identical (stiff) outer layers and a more compliant inner interlayer, perfectly bonded to one another at the interface. This structure is called three-layered plate.

The state-of-the-art review on modeling of layered structures [1–4] showed that the vast majority of research findings were related to finite element analysis [5–26] and simplified formulas based on the monolithic plate having equivalent bending properties to the layered plate [27–35]. On the contrary, research findings on analytical modeling were of the minority; moreover, they were mainly related to the interface [36–39] and the layered beam [40–48], while only few research findings were related to the layered plate [49–55].

However, analytical modeling is absolutely necessary and cannot be replaced by finite element method or simplified formulas. In fact, high values of the layer-to-interlayer elastic moduli and/or thicknesses ratios impinge on the results of finite element models and simplified formulas. Therefore, these methods call for a benchmark, which can be provided only by analytical modeling. In particular, finite element analysis results (or simplified formulas) have to be checked against exact results and the models (or the formulas) have to be calibrated to obtain the best agreement with exact solutions.

Analytical modeling of three-layered structures can be developed within the framework of either three-dimensional or two-dimensional elasticity. However, three-dimensional analytical models [46,54] are cumbersome to use. Therefore, not only can these models not be considered as viable alternatives to finite element models or simplified formulas, but also these models cannot be used as benchmarks for the other modeling approaches.

Considering this, a specific research program on the three-layered plate was started, within the framework of two-dimensional elasticity and analytical modeling. This research program obtained a model for the laminated glass plate [51] and a model for the sandwich plate [52]. Both the models assumed that each layer behaved according to the Kirchhoff–Love plate hypotheses and that the interlayer had nil elastic modulus (i.e., only the interlayer shear stiffness was considered). Additionally, [51] assumed that the interlayer was thin in comparison to the layers, which simplified the shear stress calculation.

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Nomenclature

B	side parallel to the y-axis ($0 \le y \le B$)
E	elastic modulus of the layers
E_t	in-plane elastic modulus of the interlayer
E_{tz}	out-of-plane elastic modulus of the interlayer
G	(elastic) shear modulus of the layers
G_t	(elastic) shear modulus of the interlayer
h	thickness of each layer
L	side parallel to x-axis, with $L \geqslant B$ ($0 \leqslant x \leqslant L$)
M; N	arguments of the trigonometric series ($m{\cdot}\pi/L$; $n{\cdot}\pi/B$), where m and n are odd integer positive numbers
p	lateral surface load (load per square unit of surface)
t_u; t_v	x and y components of the in-plane shear stress transferred through the upper interface, between the interlayer and the upper layer
u; v; w	components of displacements in the x, y, z directions, respectively
u_t; v_t	x and y components of the relative in-plane displacement of the upper interface with respect to the to the plate middle surface (which is immovable)
t_{vm}; v_{tm}	maximum value of t_v and v_t
w_{Max}	maximum deflection of the plate
z; z'; z''	out-of-plane axes, with origin O, O', O'' on the middle plane of the interlayer, upper layer, and lower layer, respectively
β	constitutive parameter whose value is either 1 (if $\beta_t \geq \beta_{cr}$) or 0 (if $\beta_t < \beta_{cr}$)
β_{cr}	critical value of β_t (if $\beta_t < \beta_{cr}$, the interlayer is utterly compliant, and vice versa)
β_t	interlayer stiffness that governs the distortion, defined as E_{tz}/λ
λ	half the thickness of the interlayer
ν	Poisson's ratio of the layers
ε_x; ε_y	in-plane strains in the layers and interlayer
σ_x; σ_y	in-plane normal stresses in the layers and interlayer
σ_{yi}	σ_y stresses at the upper surface of the lower layer
σ_{ym}	σ_y stresses at the lower surface of the lower layer
σ_{Max}	maximum stress value in the layered plate
σ_z	normal stress in the direction transverse to the plate
θ_{ab}	angle of rotation of the segment ab
$\zeta = h + 2 \cdot \lambda$	lever arm of the internal couple of the in-plane forces

These assumptions allowed for the obtainment of simple analytical formulations, whose application was not only easy, but even less time consuming than to generate a finite element mesh or to apply an empirical formula. Moreover, the assumptions reproduced directly the process of deformation of the layered plate, while they disregarded the mechanical behaviors that the plate was not explicitly designed to have, in order to capture the defining attributes of the layered plate. Thus, these models also provided an understanding of the phenomena involved and also represent tools for design.

Since these models were derived under the assumption of nil interlayer elastic modulus, these models provided the exact solution only for three-layered plates with utterly compliant interlayers. If the stiffness of the interlayer was non-negligible, these models did not provide the exact solution, however the solution they provided was closely approximated. Moreover, these models did not lend themselves to be used as benchmark for finite element models, since the assumption of nil interlayer elastic modulus could not be directly reproduced by using the finite element methods. In fact, the material properties of the elastic finite elements that model the interlayer have to satisfy the elasticity laws and to avoid numerical singularities.

Non-nil elastic modulus implies normal stress, σ, in the interlayer. These normal stresses σ provide the plate with extra stiffness and extra load-carrying capacity. The extra stiffness and the extra load-carrying capacity due to the in-plane σ stresses (σ_x, σ_y) are marginal, since the fibers are close to the middle plane (neutral axis). Conversely, the extra stiffness and the extra load-carrying capacity due to the out-of-plane σ stress (σ in the direction transverse to the plate, σ_z) may be significant. The greater the elastic modulus and/or thickness of the interlayer, the greater these contributions. But above all, the extra-stiffness and the extra load-carrying capacity are noticeable even for a low elastic modulus of the interlayer. Thus, results from finite element models can be compared to results from the analytical models [51,52] only if the interlayer is either very thin or described by utterly orthotropic finite elements.

This research aimed at achieving an easy-to-use analytical model like the former models [51,52], but that could also be used to check and calibrate the finite element models in a simple and straightforward manner. Thus, modeling has considered the actual value of the elastic modulus in the direction transverse to the plate. In so doing, the new model can be applied to three-layered plates with interlayer elastic modulus that is non-negligible, while the former models assumed that the interlayer elastic modulus was nil. This model uses various mathematical developments obtained for the model of the sandwich plate with discontinuous connection [56], which are referred to here.

2. System definition and modeling assumptions

The considered reference structure is the three-layered plate with cross-section formed by two isotropic layers, each one of thickness h, plus an interlayer of thickness 2λ, restrained at the boundary, and subjected to a lateral load p (Fig. 1). Global positions are identified by a Cartesian coordinate system with origin O on the middle plane of the plate, the x and y axes in the plane of the plate, and the z-axis out of the plane of the plate (Figs. 1 and 2). Local positions within the upper layer are identified by the same in-plane x and y axes, but with the out-of-plane z' axis with origin O' on the middle plane of the upper layer (and O'' and z'' for the lower layer).

The behavior of the reference structure is anti-symmetric with respect to its middle plane.

The nomenclature adopted in this paper is the same as that adopted in [52]. In particular, u_t and v_t denote the x and y components of the in-plane displacement of the upper interface with respect to the middle plane of the three-layered plate. Since anti-symmetric condition implies that the middle plane is immovable, u_t and v_t are absolute displacements. Moreover, t_u and t_v denote the x and y components of the in-plane shear stresses transferred between the interlayer and the layers through the interfaces. The positive directions of the vectors are defined in [52] and are shown in all the figures.

The modeling process within the two-dimensional elasticity framework calls for the following assumptions [51,52], which are about (1) the plate, (2) the interfaces, (3) the layers, (4) and the interlayer.