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The finite element discretized symplectic method for interface cracks

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A B S T R A C T

The method of symplectic series discretized by finite element is introduced for the stress analysis of structures having cracks at the interface of dissimilar materials. The crack is modeled by the conventional finite elements dividing into two regions: near and far fields. The unknowns in the far field are as usual. In the near field, a Hamiltonian system is established for applying the method of separable variables and the solutions are expanded in exact symplectic eigenfunctions. By performing a transformation from the large amount of finite element unknowns to a small set of coefficients of the symplectic expansion, the stress intensity factors, the displacements and stresses in the singular region are obtained simultaneously without any post-processing. The numerical results are obtained for various cracks lying at the bi-material interface, and are found to be in good agreement with the reference solutions for the interface crack problems. Some practical examples are also given.

1. Introduction

Multi-material composites and structures are being used to enhance the overall performance to take advantage of the positive attributes of the individual constituents and to minimize their weaknesses. However, premature failure due to the existence of delamination is one of the most common failure modes in composite materials and bonded joints. Therefore, the analysis of interface cracks is very important in safety investigation. Stress intensity factors (SIFs) at the tips of any interfacial cracks or flaws can be used as controlling parameters. Various numerical methods for evaluating the stress intensity factors of bi-material interface cracks have been developed.

Numerical approaches such as boundary element method (BEM) and the element free Galerkin method (EFGM) are widely used in the solution of the interface crack problems. Pan and Amadei [1] presented a boundary element formulation for the analysis of linear elastic fracture mechanics problems involving anisotropic bi-materials. Green's functions are also derived to avoid discretization along the interface except for the interfacial crack part and a special crack-tip element is introduced to capture exactly the crack-tip behavior. Matsumto et al. [2] developed a method for evaluating the SIFs of interface cracks between dissimilar materials based on the interaction energy release rates and the BEM sensitivity analysis. Hadjesfandiari and Dargush [3] developed a boundary element formulation to determine the complex SIFs associated with cracks on the interface between dissimilar materials. The oscillating stress singularity is addressed through the introduction of complex weighting functions for both displacements and tractions by non-standard numerical quadrature formulas. Pant et al. [4] evaluated the complex SIFs for bi-material interface cracks using EFGM. The material discontinuity at the interface has been modeled using a jump function with a jump parameter that governs its strength. Russo and Zuccarello [5] employed BEM to obtain the G-SIFs in the zone where the interface intersects the free edge surfaces of bonded metal-composite co-cured joints in the numerical analysis.

Many different software packages based on FEM (finite element method) techniques have been developed. The finite element method is widely used in engineering design. Nagai et al. [6,7] proposed a numerical method for evaluating SIFs of interface cracks between dissimilar anisotropic materials subjected to thermal and mechanical loads. Using the M-integral with the moving least-square method, SIFs can be automatically calculated with only the nodal displacements from the FEM. Serier et al. [8] extended the finite element method to the analysis of the behavior of an interface crack in bi-material specimen with a central hole. Noda et al. [9–11] calculated the SIFs for edge and central interface cracks in a bi-material bonded strip subjected to various loads by FEM. Ouinas et al. [12] introduced the FEM to study the performance of the bonded composite reinforcement for reducing the stress concentration at a semicircular lateral notch and for repairing cracks emanating from this kind of notch. Caner and Bažant [13] used zero-thickness interface elements to model fractures at the skin–foam interface, in both the fiber composite skins and

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the foam. Chow and Atluri [14] developed a 'mutual integral' approach combined with FEM to calculate the mixed-mode SIFs for a free-edge delamination crack in a laminate subject to tensile loading. The extended finite element method (X-FEM) for fracture mechanics was first proposed by Belytschko in 1999. Nagashima et al. [15] applied X-FEM to stress analyses of structures containing interface cracks between dissimilar materials. The interface crack can be modeled by locally changing interpolation function in the element near a crack. Belytschko and Gracie [16] presented a method for modeling dislocations in systems with arbitrary materials interfaces based on X-FEM where dislocations are modeled in the sense of Volterra. Besides the above works, the fractal geometry concept was introduced into finite element method. Leung et al. [17–19] developed a fractal finite element method (FFEM) to calculate the mixed mode stress intensity factor and the thermal effect for two-dimensional isotropic thermal crack problems. The complete eigenfunction expansion of displacement by Williams is employed for the global interpolation function. The fractal-like finite element method has been proved to be very efficient and accurate in two-dimensional static and dynamic crack problems.

In the present study, a finite element discretized symplectic method is developed for calculating the mixed mode stress intensity factors of interface cracks in multi-materials composites. The overall cracked body is divided into a finite size singular stress region near the crack tip and a regular region away from the crack tip, i.e. near field and far field. Both of the two regions are modeled by conventional FEM. In the near field, the symplectic method is employed to derive the analytical solution of the displacements and stresses. The symplectic method for solid mechanics and elasticity was first developed by Zhong and his associates [20–22]. Much research work was done on the fracture analysis [23–25]. We make use of the analytical solutions from the symplectic series and perform a displacement transformation to reduce the large number unknown nodal displacements to a small set of coefficients. The size of matrices involved is drastically reduced. We begin by establishing the general formulation of FEDSM for bi-material structures with interface cracks. Then, the symplectic method is introduced to obtain the expressions of the displacement and stress functions. Mode I and II stress intensity factors are directly obtained by some specific terms of the series. Finally, numerical comparisons to the classical solutions in literature are presented to validate the efficiency and accuracy. New results are also presented.

2. Finite element discretized symplectic method

In this section, the methodology of finite element discretized symplectic method (FEDSM) will be introduced. Denote the two dissimilar isotropic materials as M_1 and M_2. The overall interface crack is divided into near and far fields as shown in Fig. 1. The curve that separates the two fields is denoted by Γ_0. The whole structure is modeled by the conventional finite element method (FEM). The near field analytical symplectic series solutions are obtained first by symplectic method and the displacement unknowns at the nodes of the near field are then transformed to the unknown coefficients of the series while the unknowns of the far field are unaltered. It should be noted that it is not required to develop any special singular elements in the present method. Any order and any shape of finite elements can be used in the FEDSM.

In a crack problem using FEM, the static equilibrium equation is

$$\mathbf{Ku} = \mathbf{f} \tag{1}$$

where $\mathbf{K} = \begin{bmatrix} \mathbf{K}_{FF} & \mathbf{K}_{FN} \\ \mathbf{K}_{NF} & \mathbf{K}_{NN} \end{bmatrix}, \mathbf{u} = \begin{Bmatrix} \mathbf{u}_F \\ \mathbf{u}_N \end{Bmatrix}$ and $\mathbf{f} = \begin{Bmatrix} \mathbf{f}_F \\ \mathbf{f}_N \end{Bmatrix}$ are the stiffness matrix, displacement vector and force vector respectively. The subscripts F and N represent the far and near fields. The grid refinement

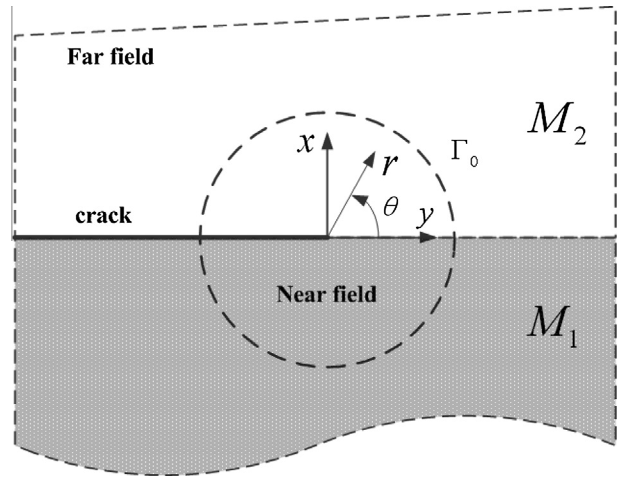

Fig. 1. Near field and far field in a cracked structure.

technique is usually used in the conventional FEM, so that high order stiffness matrix is possible. In the near field, analytical symplectic functions $\mathbf{\Phi}(r, \theta)$ in polar coordinates are first obtained by the method of separable variables in a Hamiltonian approach in the next section. In the near field, a symplectic transformation is employed to reduce the large number of unknown displacements $\mathbf{u}_N$ in Eq. (1) to a small number of unknown coefficients of symplectic eigenfunctions $\mathbf{c}$ so that $\mathbf{u}_N(r, \theta) = \mathbf{\Phi}(r, \theta)\mathbf{c}$, where $\mathbf{c}$ is the vector of unknown coefficients which is independent of the coordinates. Equivalently, at node j, the nodal displacements are evaluated at $\mathbf{u}_N(r_j, \theta_j) = \mathbf{\Phi}(r_j, \theta_j)\mathbf{c}$. Eq. (1) reduces to

$$\bar{\mathbf{K}}\bar{\mathbf{u}} = \bar{\mathbf{f}} \tag{2}$$

where $\bar{\mathbf{K}} = \begin{bmatrix} \mathbf{K}_{FF} & \mathbf{K}_{FN}\mathbf{\Phi} \\ \mathbf{\Phi}^{\mathrm{T}}\mathbf{K}_{NF} & \mathbf{\Phi}^{\mathrm{T}}\mathbf{K}_{NN}\mathbf{\Phi} \end{bmatrix}, \ \bar{\mathbf{u}} = \begin{Bmatrix} \mathbf{u}_F \\ \mathbf{c} \end{Bmatrix}, \bar{\mathbf{f}} = \begin{Bmatrix} \mathbf{f}_F \\ \mathbf{\Phi}^{\mathrm{T}}\mathbf{f}_N \end{Bmatrix}$. The unknowns now are the vector of displacements in the far field $\mathbf{u}_F$ and the vector of the handful coefficients $\mathbf{c}$ instead of the large order vector $\mathbf{u}_N$ in the singular near field. An additional advantage is that the SIFs are given explicitly in $\mathbf{c}$ and, therefore, no post-processing is required. Hence, the computational time can be reduced significantly. In addition, for multiple interface cracks, the whole domain can be divided into several sub-domains which include only one crack at a time based on the substructure method.

3. Hamiltonian system in the near field

In the near field, analytical solutions are found by the symplectic method. The interface crack located on the common edge ($\theta = 0°$) in the polar coordinate (r, θ) where the r-axis is along the radial direction with the origin located at the crack tip. Let E_1, υ_1 and E_2, υ_2 be the Young moduli and the Poisson ratio for M_1 and M_2 respectively. Denote $\partial_r = \partial/\partial r$, $\partial_\theta = \partial/\partial\theta$, the potential energy density is

$$\begin{aligned} U^i(u_r^i,\ u_\theta^i) = {} & E_i[(\partial_r u_r^i)^2 + 2\upsilon_i \partial_r u_r^i(u_r^i + \partial_\theta u_\theta^i)/r \\ & + (u_r^i + \partial_\theta u_\theta^i)^2/r^2]/[2(1-(\upsilon_i)^2)] \\ & + E^i[\partial_r u_\theta^i - (u_\theta^i - \partial_\theta u_r^i)/r]^2/[4(1+\upsilon_i)] \end{aligned} \tag{3}$$

where σ_{kl} are the components of stresses, the superscript i denotes material element 1 or 2 for the displacements u_r^i and u_θ^i. Denote $\eta = \ln r$ to eliminate the variable coefficient r in Eq. (3) and let the over-dot represent differentiation with respect to η, namely $(\dot{}) = \partial_\eta()$. In the absence of external body force, the Lagrange function in polar coordinates is $L^i(u_r^i, u_\theta^i) = r^2 U^i(u_r^i,\ u_\theta^i)$. According to references [23], the full state vector is