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# Symmetry in a problem of transverse shear of unidirectional composites

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ABSTRACT

Unidirectional fibre-reinforced composites with symmetrical structure, loaded by transverse shear, are investigated. The focus of the paper is on mathematical models for different representative cells. Transverse shear of symmetrical composites, unlike other types of loads, does not allow application of Curie's principle for detection of possible symmetry of mechanical fields. The existence of such symmetry is shown by employing the theorem proven earlier by the author. Respective boundary value problems can be formulated for the minimal representative cell. In contrast to the existing approach, which contains inaccuracy of Saint–Venant's principle, the proposed formulations are exact. It is shown that employing the symmetry cell in numerical solutions can reduce computational cost by 2–3 orders. With the use of Lagrange's and Castigliano's variational principles in generalised form, it is proven that solutions for the "infinite" cell give lower and upper bounds for the transverse shear modulus. It is proven, as well, that these bounds lie within the Voigt and Reuss bounds.



## 1. Introduction

To determine the effective properties of a composite in terms of its component properties, an infinite composite loaded in a certain way at infinity is considered. However, respective boundary value problems (BVPs) are usually solved using finite representative cells [1–8]. Those cells may differ due to the chosen definition domain of the BVP and the respective way that boundary conditions are assigned.

Let us refer to the cell as "infinite" if it occupies a finite region loaded in the same way as an infinite composite. The size of this cell has to be much larger than the characteristic length scale of inhomogeneity. The transfer of the load conditions from infinity to the "infinite" cell boundaries implies substitution of the actual shear stress field with the statically equivalent uniform stress field. Such employment of the Saint–Venant's principle contributes inaccuracy to the mathematical model. Enlargement of the cell improves accuracy but also increases computational cost.

For the composite with symmetric structure, the minimal representative cell is a periodic cell occupying the domain of the period of the stress field. Boundary conditions for the cell include conditions of periodicity and loading.

When the structure of the composite is symmetric, this symmetry can be used in two ways. First, Neumann's principle [9] allows for the detection of symmetry of anisotropy. Second, Curie's principle [10] allows, in some load cases, for the detection of mechanical fields symmetry. According to this principle, the symmetry of cause (composite structure and its load) entails a similar symmetry of effect (mechanical fields). In this case, the symmetry cell is a minimal representative cell. The cell occupies part of the periodic domain and has boundary conditions of symmetry and loading, which takes place in cases of uniaxial tension or longitudinal shear in principal directions when both the structure and the load of the composite possess planar symmetry.

Difficulties arise when such a composite is loaded with transverse shear. This problem is discussed in [5]. In this case, Curie's principle cannot detect possible symmetry of mechanical fields because one constituent of cause (namely, the structure) has symmetry planes, and another constituent (the load) is nonsymmetric (we can say, antisymmetric) with respect to these planes. However, detailed analysis [11] shows that even in such cases, a non-obvious form of mechanical fields symmetry can be present, and its use can increase the efficiency of the solution.

## 2. Description of the problem and main equations

Let us consider an infinite unidirectional composite with doubly periodic structural symmetry with periods $2a_1$ and $2a_2$ along axes $x_1$ and $x_2$. Symmetry planes of composite structure are parallel to planes $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ (Fig. 1).

The material of the constituents is isotropic or transversely isotropic with isotropic plane $x_1x_2$. The task is to calculate the effective modulus of transverse shear

$$\tilde{G}_{12} = \frac{\tilde{\tau}_{12}}{\tilde{\gamma}_{12}}, \tag{1}$$

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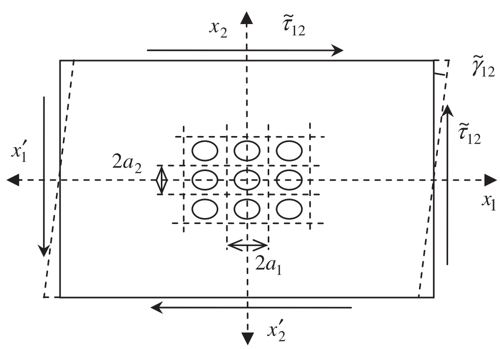


**Fig. 1.** Loading of the composite with macrostress $\tilde{\tau}_{12}$ or macrostrain $\tilde{\gamma}_{12}$ at infinity.

where $\tilde{\tau}_{12}$ and $\tilde{\gamma}_{12}$ are macrostress and macrostrain of transverse shear, respectively. Herein, all of the quantities related to macroscale are marked with a tilde. To solve the given problem, we need to specify one of the loading parameters (force loading $\tilde{\tau}_{12}$ or kinematic loading $\tilde{\gamma}_{12}$) as the boundary conditions, solve the micromechanical problem, define the micromechanical fields in a composite (displacement vector $\overline{\boldsymbol{U}}(x_1, x_2)$, strain tensor $\hat{\boldsymbol{\varepsilon}}(x_1, x_2)$ and stress tensor $\hat{\boldsymbol{\sigma}}(x_1, x_2)$), calculate another loading parameter ($\tilde{\gamma}_{12}$ or $\tilde{\tau}_{12}$) by averaging microstrains $\gamma_{12}(x_1,x_2)$ or microstresses $\tau_{12}$ $(x_1,x_2)$, and then use formula (1).

The given composite is orthotropic due to the Neumann's principle. Therefore, in case of transverse shear

$$\begin{aligned} &\tilde{\varepsilon}_3 = \varepsilon_3 = \tilde{\gamma}_{31} = \gamma_{31} = \tilde{\gamma}_{23} = \gamma_{23} = 0, \\ &\tilde{\varepsilon}_1 = \tilde{\varepsilon}_2 = 0 \end{aligned} \tag{2}$$

the problem can be classified as plane strain of piecewise uniform elastic body.

Mechanical fields in the composite are described by the generalised Hooke's law, Cauchy's equations and differential equations of equilibrium:

$$\begin{aligned} &\varepsilon_{ij} = s_{ijkl}\sigma_{kl}, \quad (i,j,k,l = 1,2,3), \\ &\varepsilon_{ij} = (U_{i,j} + U_{j,i})/2, \quad (i,j = 1,2), \\ &\sigma_{ij,j} = 0, \quad (i,j = 1,2). \end{aligned} \tag{3}$$

Boundary conditions for the force loading are given at infinity by the uniformly distributed load

$$\begin{aligned} &\tau_{12}(x_1 \to \pm\infty, x_2) \to \tilde{\tau}_{12}, \\ &\sigma_1(x_1 \to \pm\infty, x_2) \to 0, \\ &\tau_{12}(x_1, x_2 \to \pm\infty) \to \tilde{\tau}_{12}, \\ &\sigma_2(x_1, x_2 \to \pm\infty) \to 0 \end{aligned} \tag{4}$$

with complementary equations

$$\begin{aligned} &U_1(x_1 = 0, x_2 = 0) = 0, \\ &U_2(x_1 = 0, x_2 = 0) = 0, \\ &U_2(x_1 = a_1, x_2 = 0) = 0, \end{aligned} \tag{5}$$

which eliminate arbitrary deformation-free movement of the composite.

Kinematic loading at infinity is given by uniform shear strain:

$$\begin{aligned} &U_1(x_1 \to \pm\infty, x_2) \to \tilde{\gamma}_{12}x_2, \\ &U_2(x_1 \to \pm\infty, x_2) \to 0, \\ &U_1(x_1, x_2 \to \pm\infty) \to \tilde{\gamma}_{12}x_2, \\ &U_2(x_1, x_2 \to \pm\infty) \to 0. \end{aligned} \tag{6}$$

Given that elasticity parameters are discontinuous functions of coordinates, we are searching for the generalised solution of the BVP formulated above.

## 3. Existing approaches to the problem

### 3.1. Periodicity cell

Due to the periodicity of structure and, accordingly, periodicity of the mechanical fields in the composite, we can formulate the BVP for the periodicity cell $-a_1 \leqslant x_1 \leqslant a_1$, $-a_2 \leqslant x_2 \leqslant a_2$ (Fig. 2), which is equivalent to the original problem at an infinite domain.

The dimension of the cell in the $x_3$ direction here and below equals one in the measurement units of choice. Then, the boundary conditions are conditions of the periodicity of mechanical fields in view of the shear deformation:

$$\begin{aligned} &U_1(x_1 = -a_1, x_2) = U_1(x_1 = a_1, x_2), \\ &U_2(x_1 = -a_1, x_2) = U_2(x_1 = a_1, x_2), \\ &\sigma_1(x_1 = -a_1, x_2) = \sigma_1(x_1 = a_1, x_2), \\ &\tau_{12}(x_1 = -a_1, x_2) = \tau_{12}(x_1 = a_1, x_2), \\ &U_1(x_1, x_2 = -a_2) = U_1(x_1, x_2 = a_2) - 2a_2\tilde{\gamma}_{12}, \\ &U_2(x_1, x_2 = -a_2) = U_2(x_1, x_2 = a_2), \\ &\sigma_2(x_1, x_2 = -a_2) = \sigma_2(x_1, x_2 = a_2), \\ &\tau_{12}(x_1, x_2 = -a_2) = \tau_{12}(x_1, x_2 = a_2). \end{aligned} \tag{7}$$

Apart from periodicity, the first and the sixth equations take into account that transverse shear of the orthotropic material does not cause linear macrostrains ($\tilde{\varepsilon}_1 = 0, \tilde{\varepsilon}_2 = 0$). The fifth equation sets kinematic loading by macrostrain $\tilde{\gamma}_{12}$. Eq. (5) should be added.

Periodicity conditions, which are redundant to the boundary conditions

$$\begin{aligned} &\sigma_2(x_1 = -a_1, x_2) = \sigma_2(x_1 = a_1, x_2), \\ &\sigma_1(x_1, x_2 = -a_2) = \sigma_1(x_1, x_2 = a_2) \end{aligned}$$

can be employed to check the accuracy or error of the numerical solution of the BVP (3), (7) and (5).

### 3.2. "Infinite" cell

In practice, an approximate approach is used when an infinite definition domain of BVP is replaced by an "infinite" cell, consisting of the periodicity cell surrounded by $q$ layers of periodicity cells [12]. In Fig. 1, for example, a monolayer "infinite" cell is presented. In the literature, an "infinite" cell consisting of only one periodicity cell ($q = 0$) [5,13], or even only one-quarter of one [4], has been discussed and employed.

The transfer of force loading conditions from infinity to the boundaries of the "infinite" cell transforms boundary conditions (4) into the following form:

$$\begin{aligned} &\tau_{12}(x_1 = \pm l_1, x_2) = \tilde{\tau}_{12}, \\ &\sigma_1(x_1 = \pm l_1, x_2) = 0, \\ &\tau_{12}(x_1, x_2 = \pm l_2) = \tilde{\tau}_{12}, \\ &\sigma_2(x_1, x_2 = \pm l_2) = 0, \end{aligned} \tag{8}$$

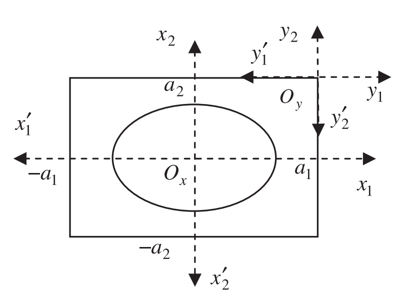


**Fig. 2.** Periodicity cell.