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Cut and join operator ring in tensor models

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Abstract

Recent advancement of rainbow tensor models based on their superintegrability (manifesting itself as the existence of an explicit expression for a generic Gaussian correlator) has allowed us to *bypass* the long-standing problem seen as the lack of eigenvalue/determinant representation needed to establish the KP/Toda integrability. As the mandatory next step, we discuss in this paper how to provide an adequate designation to each of the connected gauge-invariant operators that form a double coset, which is required to cleverly formulate a tree-algebra generalization of the Virasoro constraints. This problem goes beyond the enumeration problem *per se* tied to the permutation group, forcing us to introduce a few gauge fixing procedures to the coset. We point out that the permutation-based labeling, which has proven to be relevant for the Gaussian averages is, via interesting complexity, related to the one based on the keystone trees, whose algebra will provide the tensor counterpart of the Virasoro algebra for matrix models. Moreover, our simple analysis reveals the existence of nontrivial kernels and co-kernels for the cut operation and for the join operation respectively that prevent a straightforward construction of the non-perturbative RG-complete partition function and the identification of truly independent time variables. We demonstrate these problems by the simplest non-trivial Aristotelian **RGB** model with one complex rank-3 tensor, studying its ring of gauge-invariant operators, generated by the keystone triple with the help of four operations: addition, multiplication, cut and join.

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1. Introduction

Tensor models [1] begin to acquire attention that they deserve [2–25] as natural objects to study in the framework of the non-linear algebra [26]. In a recent series of papers [9–12], we described the technique necessary for the first step of systematic analysis of tensor models. It turned out that the simplest problem is a complete description of the Gaussian correlators, the problem which for many years remained unsolved in the case of matrix models, despite a number of brilliant insights including the celebrated Harer–Zagier formulas [27,28]. As was expected, a solution to the problem came from the synthesis of character [29] and Hurwitz [30–32] calculi (see [11]), and it appeared to be immediately generalizable to the tensor case [12,33]. Like in the matrix model case, the simplest from this perspective is the rectangular complex model of [34–36], and, among tensor models, the easiest treatable are rainbow models [9] with the highest possible "gauge" symmetry, while models with restrictions on the colorings and/or reality conditions are described by a little more complicated formulas, with the simplest example of such complications provided by the Hermitian matrix model (!). Of additional interest is the subclass of starfish rainbow models [9], where the large-N limit is automatically described by melonic diagrams (these, however, will not be considered in the present paper).

As usual in quantum field theory, the study of any such model consists of several steps: describing the symmetries and the field content of the model, enumeration and classification of operators, introduction of appropriate generating functions and evaluation of correlators/averages. Only at the last of these steps, the action/dynamics of the model is needed, though a clever choice of the generating functions to make can also depend on the action and on a particular phase of the model. Traditional analysis begins from the Gaussian phase, and then the Ward identities are used to express the correlation functions through the Gaussian spectral curve in *a functorial way* (by the procedure known as topological recursion [37]), and transition to the non-perturbative (Dijkgraaf–Vafa) phases goes through a deformation of the spectral curve. This approach is successfully developed for the one-matrix eigenvalue models (where also integrability properties are revealed and understood), and the present task is to extend it in two directions: to multi-matrix and to (multi-)tensor models. However, such extension is quite sophisticated and can hardly be made by one simple effort. As suggested in [10], we move by small steps, but in a systematic way with the hope that it will be no less straightforward for tensors than it has proven to be for matrices.

Accordingly, the very first task is to provide an efficient enumeration of operators. As was already mentioned, this step is independent of the action of the model and depends just on its field content. The problem is purely combinatorial, but one should not underestimate its significance. The choice of an appropriate language and notation is crucial for the theory of tensor models, which did not advance for years, with the main obstacle being the lack of notation like traces and determinants (while their relevant generalizations in the character/Hurwitz calculus are perfectly known within the context of non-linear algebra, see [26] and references in the last paper there). We advocate the use of permutation-group terminology, which was attempted for matrix models already in [30,31], but did not gain enough attention, both because of efficiency of other languages and, as we now understand, because of the fact that its application to the Hermitian rather than to the rainbow-like rectangular complex model (RCM) is rather clumsy. For tensor models, however, advantages of this terminology become obvious: it was actually used in the tensorial calculations in [10], and was made fully explicit in [11,12,32] and [33], where it immediately provided *generic explicit expressions* for the Gaussian correlators in arbitrary matrix and tensor models. These formulas can not be fully appreciated without detailed examples and explanations