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A geometric viewpoint on generalized hydrodynamics

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Abstract

Generalized hydrodynamics (GHD) is a large-scale theory for the dynamics of many-body integrable systems. It consists of an infinite set of conservation laws for quasi-particles traveling with effective ("dressed") velocities that depend on the local state. We show that these equations can be recast into a geometric dynamical problem. They are conservation equations with state-independent quasi-particle velocities, in a space equipped with a family of metrics, parametrized by the quasi-particles' type and speed, that depend on the local state. In the classical hard rod or soliton gas picture, these metrics measure the free length of space as perceived by quasi-particles; in the quantum picture, they weigh space with the density of states available to them. Using this geometric construction, we find a general solution to the initial value problem of GHD, in terms of a set of integral equations where time appears explicitly. These integral equations are solvable by iteration and provide an extremely efficient solution algorithm for GHD.

1. Introduction

Generalized hydrodynamics (GHD) [1,2] is a hydrodynamic theory where the notion of local equilibration to a Galilean (or relativistic) boost of a Gibbs state, is replaced by that of local relaxation to generalized Gibbs ensembles [3–5]. It is expected to emerge in appropriate hydrodynamic limits in both quantum and classical integrable many-body systems, including field theory

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and spin chains, and describes time evolution in inhomogeneous backgrounds or from inhomogeneous states. It is in the context of steady states arising from domain-wall initial conditions (see, for instance, a review [6]) that it was originally introduced, where it solved the long-standing problem of obtaining full density and current profiles in interacting integrable quantum systems [1,2]. It was generalized to include inhomogeneous force fields [7]. It is seen as emerging in integrable classical systems such as the hard rod fluid [8,9] and soliton gases [10–14], and a classical molecular dynamics solver has been developed for the general form of GHD [14]. It was applied to study spin transport [15–17], transport in the hard rod fluid [18], and quantum dynamics of density profiles such as propagating waves in interacting Bose gases [19,20]. Most of these studies concentrate on the emerging hydrodynamics at the Euler scale, at which GHD was originally formulated, but see [7,18,21] for discussions of viscosity effects. Here we do not discuss such effects.

The goal of this paper is twofold. First, we provide a geometric interpretation of GHD. We show that it is a theory for a gas of freely (inertially) propagating particles, but within a space whose metric depends both on the type and velocity of the particle, and on local distribution of particles in the gas. In the hard rod fluid, the metric has a clear interpretation: it measures the free space available between the rods, a notion that was used in [18] in a derivation of the exact solution to the domain wall initial problem. The observation that this generalizes to soliton gases, still described by GHD, then suggests the metric construction proposed here. It is worth noting some similarity, in spirit, to Einstein's theory of general relativity, where currents are conserved in a metric that is determined by the matter content. Second, we use this geometric construction in order to provide a system of integral equations that *solve the initial-value problem of GHD* in full generality. The integral equations involve the initial condition and the time parameter in an explicit fashion, essentially integrating out the time direction. They can be solved by iteration. We confirm their validity by providing comparisons with direct solutions of the GHD partial differential equations. It is surprising that a general solution to a hydrodynamic equation can be obtained, and this might connect with the integrability of the GHD equations themselves, as found in soliton gases [10–13,22].

The paper is organized as follows. In Section 2 we review some of the main features of GHD. In Section 3 we develop the geometric interpretation of GHD in generality. In Section 4, we derive the integral equations that solve the initial value problem. Finally, in Section 5 we conclude, and in Appendix A we briefly explain the specialization to the hard rod problem.

2. Overview of generalized hydrodynamics

The most powerful formulation of GHD to date [1,2] uses the physical notion of quasi-particles (although other formulations will doubtless come to the fore in the future). Integrable models solvable by Bethe ansatz, and integrable classical gases such as the hard rod model [8,9] and soliton gases [14], can be seen as models for interacting quasi-particles. A quasi-particle is specified by a spectral parameter $\theta \in \mathbb{R}$, parametrizing its momentum $p(\theta)$ (it can be taken as the velocity in the Galilean case, or the rapidity in the relativistic case; in general it is just a parametrization of the momentum). The scattering kernel $S(\theta, \alpha)$ characterizes the interaction amongst quasi-particles, and it is customary to define the differential scattering phase $\varphi(\theta, \alpha) = -\mathrm{i}\, \mathrm{d} \log S(\theta, \alpha)/\mathrm{d}\theta$, which we assume to be symmetric. In this paper we assume for simplicity that there is a single particle species, but all equations are directly generalizable to models with many species such as the Heisenberg chain, simply by viewing the rapidity θ as a multi-index $\boldsymbol{\theta} = (\theta, j)$.