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A remark on generalized complete intersections

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Abstract

We observe that an interesting method to produce non-complete intersection subvarieties, the generalized complete intersections from L. Anderson and coworkers, can be understood and made explicit by using standard Cech cohomology machinery. We include a worked example of a generalized complete intersection Calabi–Yau threefold.

0. Introduction

Calabi–Yau varieties, in particular those of dimension three, are of great interest in string theory. Since there are not many general results yet on their classification, but see [14], the explicit construction of CY threefolds is a quite important enterprise. For example, Kreuzer and Starke classified the toric fourfolds which have CY threefolds as (anticanonical) hypersurfaces [11], [3]. Besides generalizations to complete intersection CYs in certain ambient toric varieties, like products of projective spaces, there are various other examples of CY threefolds constructed with more sophisticated algebro-geometrical methods. Recent examples include [9], [7], [10].

In the recent paper [1], L. Anderson, F. Apruzzi, X. Gao, J. Gray and S-J. Lee found a very nice method to construct many more CY threefolds. The basic idea is to take a hypersurface Y in an ambient variety P and to consider hypersurfaces X in Y. These hypersurfaces need not be complete intersections in P, that is, there need not exist two sections of two line bundles on P

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whose common zero locus is X. There are various generalizations of this method, but we will stick to this basis case. As in [1], we refer to these varieties as generalized complete intersections (gCIs).

A particularly interesting and accessible case that was found and studied by Anderson and coworkers is when the ambient variety is a product of two varieties, one of which is $\mathbf{P}^1$, so $P = P_2 \times \mathbf{P}^1$. The variety P_2 they consider is a product of projective spaces, but this is not essential, one could consider any toric variety or even more general cases. The factor $\mathbf{P}^1$ is important since there are line bundles on $\mathbf{P}^1$ with non-trivial first cohomology group and this is essential to find generalized complete intersections. We review this construction in Section 1.1.

We provide a proposition, proven with standard Cech cohomology methods, that allows one, under a certain hypothesis, to find three equations (more precisely, three sections of three line bundles on P) that define X. In Section 2 we work out a detailed example, with explicit equations, of a CY threefold which was already considered in [1]. The explicit example X has an automorphism of order two and the quotient of X by the involution provides, after desingularization, another CY threefold. More generally, we think that among the gCIs found in [1] one could find more examples of CY threefolds with non-trivial automorphisms. It might be hard though to implement a systematic search as was done in [6] for complete intersection CY threefolds in products of projective spaces. We did not find new CY threefolds with small Hodge numbers (see [5] for an update on these), but the gCICY seem to be a promising class of CYs to search for these. The recent paper [4] by Berglund and Hübsch provides further techniques to deal with gCICYs whereas [2] explores string theoretical aspects of gCICYs.

1. The construction of generalized complete intersections

1.1. The general setting

Let P_2 be a projective variety of dimension n and let $P := P_2 \times \mathbf{P}^1$. We denote the projections to the factors of P by π_1, π_2 respectively. For a coherent sheaf $\mathcal{F}$ on P_2 and an integer d we define a coherent sheaf on P by:

$$\mathcal{F}[d] := \pi_1^*\mathcal{F} \otimes \pi_2^*\mathcal{O}_{\mathbf{P}^1}(d) .$$

The Künneth formula gives

$$H^r(P, \mathcal{F}[d]) = \oplus_{p+q=r} H^p(P_2, \mathcal{F}) \otimes H^q(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d)) .$$

Recall that the only non-zero cohomology of $\mathcal{O}_{\mathbf{P}^1}(d)$ is: $h^0(\mathcal{O}_{\mathbf{P}^1}(d)) = h^1(\mathcal{O}_{\mathbf{P}^1}(-2-d)) = d+1$ for $d \geq 0$ and a basis for $H^0(\mathcal{O}_{\mathbf{P}^1}(d))$ is given by the monomials $z_0^i z_1^{d-i}$, $i = 0, \ldots, d$, where $(z_0 : z_1)$ are the homogeneous coordinates on $\mathbf{P}^1$.

Let L be a line bundle on P_2 and assume that $L[d]$, for some $d \geq 1$, has a non-trivial global section F. Using the Künneth formula, we can write $F = \sum_i f_i z_0^i z_1^{d-i}$ for certain sections $f_i \in H^0(P_2, L)$. Let $Y = (F)$ be the zero locus of F in P. We assume that Y is a (reduced, irreducible) variety, although this will not be essential in this section.

To define a codimension two subvariety of P, we consider another line bundle M on P_2. The Künneth formula shows that $M[-e]$ has no global sections if $e \geq 1$. But upon restricting to Y, the vector space $H^0(Y, M[-e]_{|Y})$ could still be non-trivial. In fact, from the exact sequence

$$0 \longrightarrow (L^{-1} \otimes M)[-d-e] \stackrel{F}{\longrightarrow} M[-e] \longrightarrow M[-e]_{|Y} \longrightarrow 0 \tag{1}$$