# Universal correlations in the nuclear symmetry energy, slope parameter, and curvature

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A B S T R A C T

From general Fermi liquid theory arguments, we derive correlations among the symmetry energy ($J$), its slope parameter ($L$), and curvature ($K_{\mathrm{sym}}$) at nuclear matter saturation density. We argue that certain properties of these correlations do not depend on details of the nuclear forces used in the calculation. We derive as well a global parametrization of the density dependence of the symmetry energy that we show is more reliable, especially at low densities, than the usual Taylor series expansion around saturation density. We then benchmark these predictions against explicit results from chiral effective field theory.

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The nuclear isospin-asymmetry energy, which characterizes the energy cost of converting protons into neutrons in an interacting many-body system, is an important organizing concept linking the properties of atomic nuclei to the structure and dynamics of neutron stars. In particular the isospin-asymmetry energy governs the proton fraction of dense matter in beta equilibrium, the thickness of neutron star crusts, and the typical radii of neutron stars [1–6]. For these reasons the nuclear isospin-asymmetry energy is a primary focus of experimental investigations at current and next-generation rare-isotope facilities such as the Radioactive Isotope Beam Factory (RIBF), the Facility for Antiproton and Ion Research (FAIR), and the Facility for Rare Isotope Beams (FRIB).

In recent years, theoretical [6–10] and experimental [11–15] studies have reduced the uncertainties on the isospin-asymmetry energy at and below the density scales of normal nuclei, but more challenging is to derive constraints at the higher densities reached in the cores of neutron stars. Given the experimental difficulties of creating and studying high-density, low-temperature matter in the lab, an alternative strategy has been to extract the coefficients in the Taylor series expansion of the isospin-asymmetry energy about nuclear matter saturation density. For instance, the slope parameter has been shown to correlate strongly with neutron skin thicknesses in nuclei [13,16,17], nuclear electric dipole polarizabilities [18–23], and the difference in charge radii of mirror nuclei [24,25]. Determining the isospin-asymmetry energy curvature is more challenging [26–28] with larger associated uncertainties.

A feature observed in many experimental and theoretical investigations is a nearly linear correlation between the value of the isospin-asymmetry energy at nuclear matter saturation density, its slope, and curvature (for a recent comprehensive analysis, see Ref. [29]). In Ref. [10] it was shown that even chiral nuclear potentials at next-to-leading order (NLO), which are rather simplistic and contain no three-body forces, exhibit a correlation slope consistent with previous microscopic calculations at N2LO and N3LO in the chiral expansion. This suggests that certain aspects of the correlation are ultimately associated with low-energy physics well described even at next-to-leading order in the chiral expansion. In the present work we will demonstrate that this is indeed the case and that the slope of the correlation can be derived without referring to detailed properties of the nucleon–nucleon potential. The overall scale is then set by a few constants that can in principle be extracted from the properties of low-density homogeneous matter. We also show that the same arguments can be used to derive the slope of the correlation between the symmetry energy and its curvature at nuclear matter saturation density.

We take as a starting point the nuclear symmetry energy $S(\rho)$, which is defined as the difference in the energy per nucleon between neutron matter and symmetric nuclear matter at a given density:

$$S(\rho) = \frac{E}{N}(\rho, \delta_{np} = 1) - \frac{E}{N}(\rho, \delta_{np} = 0), \qquad (1)$$

where $\rho$ is the total baryon number density, $N$ is the total baryon number, and $\delta_{np} = (\rho_n - \rho_p)/(\rho_n + \rho_p)$ is the isospin asymmetry

parameter. A Maclaurin expansion of the nuclear equation of state around symmetric nuclear matter

$$\frac{E}{N}(\rho, \delta_{np}) = \sum_{n=0}^{\infty} A_{2n}(\rho) \delta_{np}^{2n} \tag{2}$$

is in general [30,31] nonconvergent due to the appearance of nonanalytic logarithm terms that appear beyond a mean field description of the nuclear equation of state:

$$\frac{E}{N}(\rho, \delta_{np}) = A_0(\rho) + S_2(\rho)\delta_{np}^2 + \sum_{n=2}^{\infty} (S_{2n} + L_{2n} \ln |\delta_{np}|) \delta_{np}^{2n}. \tag{3}$$

However, all contributions beyond the quadratic term $S_2$, which we call the isospin-asymmetry energy, have been shown to be small at low temperatures when realistic nuclear forces are employed [31–37]. It is therefore a good approximation to identify the nuclear symmetry energy $S(\rho)$ with the isospin-asymmetry energy $S_2(\rho)$. In practice it is the latter quantity that can be inferred from laboratory measurements of finite nuclei.

It is common to expand the density dependence of the isospin-asymmetry energy in a Taylor series around nuclear saturation:

$$S_2(\rho) = J + L\left(\frac{\rho - \rho_0}{3\rho_0}\right) + \frac{1}{2} K_{sym}\left(\frac{\rho - \rho_0}{3\rho_0}\right)^2 + \cdots, \tag{4}$$

where $J \equiv S_2(\rho_0)$. The parameters $L$ and $K_{sym}$ can then be extracted from properties of finite nuclei [38], but their uncertainties are much larger than that associated with the isospin-asymmetry energy. For this reason correlations between the nuclear symmetry energy, the slope parameter, and curvature are routinely investigated within a range of theoretical models [7,26–29,39–41].

In Fermi liquid theory the nuclear isospin-asymmetry energy is rigorously defined at all densities in terms of the isotropic component $f_0'$ of the isovector-scalar quasiparticle interaction according to

$$S_2(\rho) = \frac{k_F^2}{6m^*}\left(1 + \frac{2m^*k_F}{\pi^2} f_0'\right), \tag{5}$$

where $k_F$ is the Fermi momentum and the nucleon effective mass $m^*$ is related to the Fermi liquid parameter $f_1$ through

$$\frac{1}{m^*} = \frac{1}{m} - \frac{2k_F}{3\pi^2} f_1. \tag{6}$$

These Fermi liquid parameters are obtained by performing a Legendre polynomial decomposition of the central quasiparticle interaction

$$\mathcal{F}(\vec{p}_1, \vec{p}_2) = \sum_l \left(f_l + g_l \vec{\sigma}_1 \cdot \vec{\sigma}_2 + f_l' \vec{\tau}_1 \cdot \vec{\tau}_2 \right.$$
$$\left. + g_l' \vec{\sigma}_1 \cdot \vec{\sigma}_2 \vec{\tau}_1 \cdot \vec{\tau}_2\right) P_l(\cos\theta), \tag{7}$$

in the relative angle $\cos\theta = \hat{p}_1 \cdot \hat{p}_2$. Expanding Eq. (5) we obtain

$$S_2(\rho) = \frac{k_F^2}{6m} + \frac{k_F^3}{9\pi^2}\left[3f_0'(\rho) - f_1(\rho)\right]. \tag{8}$$

The relationship in Eq. (8) can be used to derive a correlation between the symmetry energy slope parameter $L$ and the symmetry energy $J$ at nuclear matter saturation density:

$$L = 3\rho_0 \left.\frac{dS_2}{d\rho}\right|_{\rho_0} = 3J - S_0 + \frac{\rho_0}{6}\left(3k_F \frac{df_0'}{dk_F} - k_F \frac{df_1}{dk_F}\right)\bigg|_{k_F^0}, \tag{9}$$

where $\rho_0 = 0.16\,\text{fm}^{-3}$ is nuclear matter saturation density, $k_F^0$ is the Fermi momentum at saturation density, and $S_0 \equiv (k_F^0)^2/(6m)$

is the noninteracting part of the isospin-asymmetry energy at saturation density. To study the additional correlation between $L$ and $J$ associated with the density derivative terms on the right-hand side of Eq. (12), we perform a Taylor series expansion of the quantity $3f_0' - f_1$ around a small reference density set by $k_F = k_r$:

$$3f_0'(k_F) - f_1(k_F) = a_0 + a_1 \beta + \frac{1}{2}a_2 \beta^2 + \cdots, \tag{10}$$

where $\beta = (k_F - k_r)/k_r$.

In principle the choice of reference Fermi momentum $k_r$ is arbitrary, but we note several constraints. First, logarithmic contributions to the isospin-asymmetry energy of the form $\log(1 + 4k_F^2/m_\pi^2)$ arise from the one-pion-exchange Fock diagram [42] and formally require that $k_r \gtrsim 0.9\,\text{fm}^{-1}$ in order for the Taylor series to be convergent at nuclear matter saturation density. Second, $k_r$ should be small enough that a reliable calculation of the Fermi liquid parameters $f_0'(k_r)$ and $f_1(k_r)$ may be achieved within chiral effective field theory (EFT) whose formal expansion parameter at the reference density would be $k_r/\Lambda_\chi$, where $\Lambda_\chi \sim 500\,\text{MeV}$ is a typical momentum-space cutoff scale in realistic chiral nucleon–nucleon potentials. The effect of the less certain three-body forces, which start contributing to the homogeneous matter equation of state at a density $\rho \simeq 0.03\,\text{fm}^{-3}$ [43–45], should also be minimized. From these considerations we choose the reference Fermi momentum $k_r = 0.9\,\text{fm}^{-1} = 175\,\text{MeV}$, which satisfies $(k_F^0 - k_r)/k_r \simeq 0.5$, where $k_F^0$ is the Fermi momentum of nuclear matter at saturation density. We will show later that this value, which corresponds to the density $\rho \simeq 0.05\,\text{fm}^{-3}$, is large enough that keeping the first three terms in the Taylor series expansion in Eq. (10) provides a good description of the isospin-asymmetry energy somewhat above nuclear saturation density, even though formally the series fails to converge in that regime.

Only the terms in Eq. (10) proportional to $a_1$ and $a_2$ result in additional correlations between $L$ and $J$. Our strategy is therefore to absorb the dependence of the $L$ vs. $J$ correlation on the choice of nuclear interaction into the coefficient $a_0$, which we expect to be very similar for different nuclear force models. Combining Eqs. (10) and (12) and defining $\gamma \equiv 2k_F^2/(k_F^2 - k_r^2)$, we find the unique solution

$$L = (3 + \gamma)J - (1 + \gamma)S_0 - \gamma \frac{\rho_0}{6}(a_0 - \eta_1 a_1 + \eta_1 a_2), \tag{11}$$

where $\eta_1 = (\beta + 1)\gamma^{-1} - \beta$. This rearrangement simultaneously minimizes the residual importance of $a_1$ and $a_2$ after we have included their effect on the $L$ vs. $J$ correlation through the term $(3 + \gamma)J$. For instance, if $k_r = 0.9\,\text{fm}^{-1}$, then $\gamma \simeq 3.7$ and $\eta_1 \simeq -0.08$. We therefore expect $a_0$ to give the dominant contribution to the last term on the right-hand side of Eq. (11).

In Fig. 1 we show the values of the symmetry energy and its slope parameter at nuclear matter saturation density for a set of seven realistic nuclear potentials [46–49] at different orders in the chiral expansion and different values of the momentum-space cutoff $\Lambda$: {N1LO_450, N1LO_500, N2LO_450, N2LO_500, N3LO_414, N3LO_450, N3LO_500}. Since the equation of state of pure neutron matter is well converged in many-body perturbation theory up to nuclear matter saturation density using modern chiral effective field theory forces with $\Lambda \lesssim 500\,\text{MeV}$ [10], we compute in Fig. 1 the symmetry energy assuming a symmetric nuclear matter saturation density $\rho = 0.155$–$0.165\,\text{fm}^{-3}$ and a saturation binding energy per particle of 16 MeV. The error bars on the data points arise from varying the saturation density within the above range. We can also extract the isospin-asymmetry energy directly from a calculation of the quasiparticle interaction in Fermi liquid theory starting from chiral two- and three-body forces as described in