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Exact gravitational plane waves and two-dimensional gravity

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A B S T R A C T

We discuss dynamical aspects of gravitational plane waves in Einstein theory with massless scalar fields. The general analytic solution describes colliding gravitational waves with constant polarization, which interact with scalar waves and, for generic initial data, produce a spacetime singularity at the focusing hypersurface. There is, in addition, an infinite family of regular solutions and an intriguing static geometry supported by scalar fields. Upon dimensional reduction, the theory can be viewed as an exactly solvable two-dimensional gravity model. This provides a new viewpoint on the gravitational dynamics. Finally, we comment on a simple mechanism by which short-distance corrections in the two-dimensional model can remove the singularity.

An important problem in General Relativity (GR) is understanding detailed aspects of its highly non-linear dynamics and the time evolution of general distributions of matter. This includes, in particular, problems involving the emergence of spacetime singularities, gravitational plane wave collisions and black hole formation. A complete account of back-reaction in general time-dependent configurations is typically very complicated, but there are cases where the exact solution is known.

Since the classic works by Einstein and Rosen [1] and Bondi, Pirani and Robinson [2], many examples of exact gravitational plane waves have been discovered in the literature and investigated in great detail (see *e.g.* [3–10] and references therein). Plane waves play an important role in physics and, recently, the LIGO experiment led to a major breakthrough by the direct detection of gravitational waves. Unlike electromagnetic or sound waves, because of the non-linearity of general relativity one cannot just add individual gravitational waves to build up the resulting geometry. This note is aimed to discuss a new perspective on the problem.

Our starting point is Einstein gravity coupled to free massless scalar fields,

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-G}\left(R^{(4)} - 2\partial_\mu\phi^I\partial^\mu\phi^I\right) , \qquad I = 1, ..., N .$$

Note that we are using a non-canonical normalization for the scalar fields. Next, we consider the ansatz

$$ds^2 = g_{ij}(x,t)dx^i dx^j + e^{-2\Phi(x,t)}(e^{2\varphi}dy^2 + e^{-2\varphi}dz^2) ,$$
$$\phi^I = \phi^I(x,t) .$$

The effective two-dimensional action is given by

$$S = \int d^2x\sqrt{-g}e^{-2\Phi}\left(R^{(2)} + 2\partial_i\Phi\partial^i\Phi - 2\partial_i\varphi\partial^i\varphi - 2\partial_i\phi^I\partial^i\phi^I\right) . \tag{1}$$

The solutions of this two-dimensional field theory thus describe the gravitational interaction between four-dimensional plane waves which at the same time interact with massless matter. In particular, the plane waves may collide and gravitational collapse, in certain cases forming Killing–Cauchy horizons.

The vacuum solutions are well known. Our new results will include the construction of the most general solution to the system in the presence of a non-zero stress tensor (including the backreaction) and the interpretation of the gravitational dynamics as scattering processes in two-dimensional gravity. In addition, we will identify a new class of four-dimensional geometries supported by massless scalars free of curvature singularities, which have apparently escaped the notice of earlier investigations.

In contrast to other exactly solvable gravitational models in two dimensions, the present model originates from Einstein four-dimensional gravity; for every phenomenon taking place in the two-dimensional world there is a corresponding physical phenomenon in four dimensions. Note that other dimensional reduction ansätze from Einstein theory do not lead to a solvable model. For example, the Einstein–Hilbert action minimally coupled

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to scalar fields and restricted to spherically symmetric configurations leads to equations that cannot be solved in general form.

It is convenient to choose the conformal gauge where the metric takes the familiar form,

$$ds^2 = -e^{2\sigma(x^+,x^-)}dx^+dx^- + e^{-2\Phi(x^+,x^-)}(e^{2\varphi}dy^2 + e^{-2\varphi}dz^2)\ ,$$
$$\phi^I = \phi^I(x^+,x^-)\ , \tag{2}$$

with $x^\pm = t \pm x$. Except for the scalar fields, these are the gravitational waves studied by Szekeres [4] and a particular case of Einstein–Rosen waves admitting Killing vectors ∂_y and ∂_z. One has a cylindrical wave upon interpreting y as azimuth angle and x as radial coordinate. Using $R^{(2)} = 8e^{-2\sigma}\partial_+\partial_-\sigma$, we find

$$S = 2\int dx^+dx^- e^{-2\Phi}\left(2\partial_+\partial_-\sigma - 2\partial_+\Phi\partial_-\Phi + 2\partial_+\phi^J\partial_-\phi^J\right)\ . \tag{3}$$

Here $J = 1, ..., N+1$, with $\phi^{N+1} \equiv \varphi$. By the following field redefinition,

$$u = e^{-2\Phi}\ , \qquad v = -\sigma + \frac{\Phi}{2}\ , \tag{4}$$

the action takes the form

$$S = 4\int dx^+dx^-\ \left(\partial_+u\partial_-v + u\partial_+\phi^J\partial_-\phi^J\right)\ . \tag{5}$$

The action (5) can alternatively be viewed as a 2d non-linear σ-model, where the $N+3$-dimensional target metric represents an homogeneous pp-wave in Baldwin–Jeffery–Rosen coordinates (a detailed discussion on this two-dimensional σ-model and its generalizations can be found in [11]). A string σ-model with action (5) and its connection with two-dimensional gravity first appeared in [12]. Here we are interested in the connection between (5) and four-dimensional Einstein gravity, which was not noticed before, and in aspects of the four-dimensional exact analytic solutions.

In terms of $u,\ v$, the four-dimensional spacetime metric is

$$ds^2 = -\frac{e^{-2v}}{\sqrt{u}}\,dx^+dx^- + u(e^{2\varphi}dy^2 + e^{-2\varphi}dz^2)\ .$$

The key point of the solvability of the model is that the equation of motion for v implies

$$\partial_+\partial_-u = 0 \ \longrightarrow\ u = u_+(x^+) + u_-(x^-)\ .$$

The remaining equations and constraints (coming from the g_{++} and g_{--} equations) are

$$\partial_+\partial_-v = \partial_+\phi^J\partial_-\phi^J\ , \tag{6}$$
$$\partial_+(u\partial_-\phi^J) + \partial_-(u\partial_+\phi^J) = 0\ ,$$
$$\partial_\pm^2 u + 2\partial_\pm u\partial_\pm v + 2u\partial_\pm\phi^J\partial_\pm\phi^J = 0\ .$$

By a conformal transformation we can choose u to be linear in the coordinates. This basically leads to three possible choices (with different physics). The choice $u = x^+$ leads to pp-waves with fields depending only on x^+. The choices $u = 2x$ or $u = 2t$ involve more interesting physics. In particular, $u = 2t$ includes solutions representing the formation of Milne horizons by colliding gravitational waves. The case $u = 2x$ includes domain wall singularities, cylindrical gravitational waves and other types of traveling waves. The solutions in the two cases $u = 2x$ or $u = 2t$ are formally connected by suitably renaming coordinates. For concreteness, let us begin with the case $u = 2x$. The equations of motion take the form

$$x(\partial_t^2 - \partial_x^2)\phi^J - \partial_x\phi^J = 0\ , \tag{7}$$
$$(\partial_t^2 - \partial_x^2)v = \partial_t\phi^J\partial_t\phi^J - \partial_x\phi^J\partial_x\phi^J\ , \tag{8}$$

and the constraints become

$$\partial_t v + 2x\partial_t\phi^J\partial_x\phi^J = 0\ , \tag{9}$$
$$\partial_x v + x\left(\partial_t\phi^J\partial_t\phi^J + \partial_x\phi^J\partial_x\phi^J\right) = 0\ . \tag{10}$$

Note that we have three equations for v. It is easy to check that (9) and (10) are integrable and that they imply equation (8).

The general solution to the classical equations of motion including matter is readily found by introducing the Fourier transform (we omit the superindex J)

$$\phi(x,t) = b_0\ln x + \int_{-\infty}^{\infty} d\omega\ e^{i\omega t}\tilde\phi(\omega,x)\ . \tag{11}$$

Thus $\tilde\phi$ obeys

$$x(\omega^2 + \partial_x^2)\tilde\phi + \partial_x\tilde\phi = 0\ ,$$

with general solution

$$\tilde\phi(\omega,t) = a(\omega)\ J_0(\omega x) + b(\omega)\ Y_0(\omega x)\ , \tag{12}$$

where J_0 and Y_0 are Bessel functions. Then, $v(x,t)$ is easily obtained by integration using (10). This is a simple integral that leads to a quadratic form in Bessel functions, as shown below in an example. This describes all solutions in GR of the form (2) (adding to the family the similar solutions with $u = 2t$ and the pp-waves with $u = x^+$).

The solutions with $\phi^I = 0$ are well known. A classic example is [4]

$$\varphi = \frac{1}{\sqrt{x^2-t^2}}\ , \qquad v = \frac{x^2}{2(x^2-t^2)^2}\ , \qquad \phi^I = 0\ , I = 1, ..., N\ .$$

It is easy to check that it solves (7)–(10). It can also be obtained by substituting $a(\omega) = \text{const.}$, $b(\omega) = 0$, $b_0 = 0$, into the general solution (11), (12).

One can study axisymmetric gravitational waves by renaming $x \to r$, $y \to \theta$, with (r,θ,z) representing standard cylindrical coordinates. Four-dimensional Minkowski spacetime, $ds^2 = -dt^2 + dr^2 + r^2d\theta^2 + dz^2$, arises from the vacuum solution $u = 2r$, $\varphi = \frac{1}{2}\ln\frac{r}{2}$, $v = -\frac{1}{4}\ln 2r$. On the other hand, with the choice $u = 2t$, the analogous solution describes a Milne space, $ds^2 = -dt^2 + dx^2 + t^2dy^2 + dz^2$. These geometries will describe the geometry of colliding plane waves near the focusing hypersurface for a specific choice of initial data leading to a spacetime free from curvature singularities.

A particular case of the general solution (12) corresponds to an axisymmetric wave discussed by Kramer [9]. This is the vacuum solution $\varphi = \frac{1}{2}\ln\frac{r}{2} + a_0 J_0(\omega_0 x)\cos\omega_0 t$. This is a time-dependent solution which is regular at $r = 0$. At this point, the Riemann-squared curvature invariant is a periodic function of t. A more general class of regular vacuum solutions is found by setting

$$\varphi = \frac{1}{2}\ln\frac{r}{2} + \int d\omega\, a(\omega) J_0(\omega r)\cos\omega t. \tag{13}$$

They represent standing cylindrical waves that extend the Kramer solution to more general wave profiles. Another interesting solution is a standing (but time-dependent) gravitational wave found in [13] for Einstein theory coupled to a single massless scalar. The metric has a domain wall singularity.