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Gravitational waves in conformal gravity

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ABSTRACT

We consider the gravitational radiation in conformal gravity theory. We perturb the metric from flat Mikowski space and obtain the wave equation after introducing the appropriate transformation for perturbation. We derive the effective energy-momentum tensor for the gravitational radiation, which can be used to determine the energy carried by gravitational waves.

1. Introduction

The detection of gravitational waves (GWs) by the LIGO Collaboration is a milestone in GW research and opens a new window to probe general relativity (GR) and astrophysics [1–4]. Future space-borne detectors will offer access to an unprecedented signal sensitivity [5], thus it is worthwhile to explore GWs in alternative theories of gravity. Gravitational wave were considered in $f(R)$ theories [6–17], in scalar-tensor theories [18–20], in $f(T)$ theories [21] and in fourth-order gravity [22]. The evolution equation for gravitational perturbation in four dimensional spacetime in presence of a spatial extra dimension has been derived in [23]. The linear perturbation of higher-order gravities has been discussed in [24]. Following the original work by Weyl [25] (for review, see [26]), conformal gravity (CG), as a possible candidate alternative to GR, attracts much attention. It can give rise to an accelerated expansion [27]. It was tested with astrophysical observations and had been confirmed that it does not suffer from an age problem [28]. It can describe the rotation curves of galaxies without dark matter [29]. Cosmological perturbations in CG were investigated in [30,31]. The particle content of linearized conformal gravity was considered in [32]. It had been shown that CG accommodates well with currently available SNIa and GRB samples [33–35]. A series of dynamical solutions in CG were found in [36]. Mass decomposition of the lens galaxies of the Sloan Lens Advanced Camera for Surveys in CG was discussed in [37]. Recently it was indicated that conformal gravity can potentially test well against all astrophysical observations to date [38]. It has been shown that CG can also give rise to an inflationary phase [39]. The holographic two-point functions of four dimensional conformal gravity was computed in [40]. It was shown that four dimensional conformal gravity plus a simple Neumann boundary condition can be used to get the semiclassical (or tree level) wavefunction of the universe of four dimensional asymptotically de-Sitter or Euclidean anti-de Sitter spacetimes [41]. A simple derivation of the equivalence between Einstein and Conformal Gravity (CG) with Neumann boundary conditions was provided in [42]. It was argued that Weyl action should be added to the Einstein–Hilbert action [43].

CG is also confronted with some challenges. It has been shown that CG does not agree with the predictions of general relativity in the limit of weak fields and slow motions, and it is therefore ruled out by Solar System observations [44]. It suggested that without dark matter CG can not explain the properties of X-ray galaxy clusters [45]. It is not able to describe the phenomenology of gravitational lensing [46]. The cosmological models derived from CG are not likely to reproduce the observational properties of our Universe [47].

In this paper, we will consider gravitational radiation in CG. We aim to find the equations of gravitational radiation and the energy-momentum tensor of the GWs. These results will be valuable for future observations of GWs to test gravity theories alternative to GR.

This paper is organised as follows. We begin with a review of the CG theory. In Section 3, we consider GWs in CG. In Section 4, we will discuss the energy-momentum tensor of the GWs. Finally, we will briefly summarize and discuss our results.

2. Basic equations for conformal gravity

Besides of the general coordinate invariance and equivalence principle structure of general relativity, CG possesses an additional local conformal symmetry in which the action is invariant under

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local conformal transformations on the metric: $g_{\mu\nu} \to e^{2\alpha(x)} g_{\mu\nu}$. This symmetry forbids the presence of any $\Lambda\sqrt{-g}d^4x$ term in the action, so CG does not suffer from the cosmological constant problem [48]. Under such a symmetry, the action of CG in vacuum is given by

$$\begin{aligned}\mathcal{I} &= -\alpha_g \int C_{\mu\nu\kappa\lambda} C^{\mu\nu\kappa\lambda} \sqrt{-g} d^4x \\ &= -\alpha_g \int \left[R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2 \right] \sqrt{-g} d^4x \\ &= -2\alpha_g \int \left[R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2 \right] \sqrt{-g} d^4x,\end{aligned} \tag{1}$$

where $C_{\mu\nu\kappa\lambda}$ the Weyl tensor and α_g is a dimensionless coupling constant, unlike general relativity. To obtain the last equation, we have taken into account the fact that the Gauss–Bonnet term, $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2$, is a total derivative in four-dimension spacetime. Here we take the signature of the metric as $(-,+,+,+)$ and the Rieman tensor is defined as $R^{\lambda}{}_{\mu\nu\kappa} = \partial_\kappa \Gamma^\lambda{}_{\mu\nu} - \partial_\nu \Gamma^\lambda{}_{\mu\kappa} + \Gamma^\alpha{}_{\mu\nu}\Gamma^\lambda{}_{\alpha\kappa} - \Gamma^\alpha{}_{\mu\kappa}\Gamma^\lambda{}_{\alpha\nu}$ from which the Ricci tensor is obtained $R_{\mu\nu} = g^{\alpha\beta}R_{\alpha\mu\beta\nu}$, and the Ricci scalar is defined as $R = g^{\mu\nu}R_{\mu\nu}$. If to include the matter in the action, the energy-momentum tensor of matter should also be Weyl invariant. Due to the conformal invariance one can choose a gauge in which the scalar field is constant. Then it is obvious that the Ricci scalar term belongs to the gravity part of the theory and cannot be discarded in the vacuum case. The only way this can be achieved is by choosing the vacuum expectation value of the scalar as zero. But this means that the fermion mass term also vanishes, which means that one can consider only massless matter with this theory. Hence, one should either say that one only considers massless matter or one has to take the Ricci term into account (for details see [49]) in CG. The most general local matter action for a generic scalar and spinor field coupled conformally to gravity has been proposed in [49]. Here we focus on the vacuum case corresponding to conformal gravity with massless matter. Variation with respect to the metric generates the field equations

$$4\alpha_g W_{\mu\nu} = 0, \tag{2}$$

where

$$\begin{aligned}W_{\mu\nu} = &-\frac{1}{6}g_{\mu\nu}R^{;\lambda}{}_{;\lambda} + \frac{2}{3}R_{;\mu;\nu} + R_{\mu\nu;\lambda}{}^{;\lambda} - R_{\lambda\nu;\mu}{}^{;\lambda} - R_{\lambda\mu;\nu}{}^{;\lambda} \\ &+ \frac{2}{3}RR_{\mu\nu} - 2R_{\mu\lambda}R_\nu{}^\lambda + \frac{1}{2}g_{\mu\nu}R_{\lambda\kappa}R^{\lambda\kappa} - \frac{1}{6}g_{\mu\nu}R^2.\end{aligned} \tag{3}$$

Since $W^{\mu\nu}$ is obtained from an action that is both conformal invariant and general coordinate invariant, it is traceless and kinematically covariantly conserved: $W^\mu{}_\mu \equiv g_{\mu\nu}W^{\mu\nu} = 0$ and $W^{\mu\nu}{}_{;\nu} = 0$.

3. Gravitational waves in conformal gravity

Here we are interested in vacuum GWs of CG. Recently the GWs of CG with matter are discussed in [50], however, the results in vacuum case cannot be derived simply from the non-vacuum case by letting the energy-momentum tensor as zero. The linearized framework provides a natural way to study gravitational waves, which is a weak-field approximation that assumes small deviations from a flat background

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{4}$$

where $|h_{\mu\nu}| \sim \epsilon$ which is a small parameter. We will consider terms up to $\mathcal{O}(\epsilon)$. Thus the inverse metric is $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ where the indices are raised by used the Minkowski metric. To the first-order, the covariant derivative of any perturbed quantity will be the same as the partial derivative, so the connection and the Riemann tensor are, respectively, given by

$$\Gamma^{(1)\rho}{}_{\mu\nu} = \frac{1}{2}\eta^{\rho\lambda}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}), \tag{5}$$

$$R^{(1)\lambda}{}_{\mu\nu\rho} = \frac{1}{2}(\partial_\mu\partial_\rho h^\lambda{}_\nu + \partial^\lambda\partial_\nu h_{\mu\rho} - \partial_\mu\partial_\nu h^\lambda{}_\rho - \partial^\lambda\partial_\rho h_{\mu\nu}). \tag{6}$$

Contracting the Riemann tensor gives the Ricci tensor

$$R^{(1)}{}_{\mu\nu} = \frac{1}{2}(\Box h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial_\mu\partial_\lambda h^\lambda{}_\nu - \partial_\nu\partial_\lambda h^\lambda{}_\mu), \tag{7}$$

where the d'Alembertian operator is $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$. Contracting the Ricci tensor gives the first-order Ricci scalar

$$R^{(1)} = \Box h - \partial_\mu\partial_\nu h^{\mu\nu}. \tag{8}$$

Inserting Eqs. (7) and (8) into (3) and retaining terms to the first-order, we obtain

$$W^{(1)}{}_{\mu\nu} = -\frac{1}{6}\eta_{\mu\nu}\Box R^{(1)} + \frac{2}{3}R^{(1)}_{,\mu\nu} + \Box R^{(1)}_{\mu\nu} - R^{(1)}_{\lambda\nu,\mu}{}^{,\lambda} - R^{(1)}_{\lambda\mu,\nu}{}^{,\lambda}. \tag{9}$$

In general relativity, if we define the trace-reversed perturbation $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ and impose the Lorenz gauge $\partial^\mu\bar{h}_{\mu\nu} = 0$, the linearized vacuum Einstein field equations reduce to the wave equation

$$\Box\bar{h}_{\mu\nu} = 0. \tag{10}$$

We can apply this similar standard reasoning within the CG framework and find a quantity $\bar{h}_{\mu\nu}$ that satisfies a wave equation when linearizing the field equations (3). The rank-two tensors in linearized conformal gravity are: $h_{\mu\nu}$, $\eta_{\mu\nu}$, $R^{(1)}_{\mu\nu}$, and $\partial_\mu\partial_\nu$. In order to eliminate $R^{(1)}_{\mu\nu}$, we will try the simper combination $\eta_{\mu\nu}R^{(1)}$. The linearized field equations (3) is forth-order, we hope to get second-order wave equations which can be easily solved, so we look for a solution with the following form

$$\bar{h}_{\mu\nu} = \Box h_{\mu\nu} + \alpha\eta_{\mu\nu}\Box h + \beta\eta_{\mu\nu}R^{(1)}, \tag{11}$$

where α and β are constants. Taking the trace of Eq. (11) yields

$$\bar{h} = (4\alpha + 1)\Box h + 4\beta R^{(1)}. \tag{12}$$

So we can eliminate $h_{\mu\nu}$ in terms of $\bar{h}_{\mu\nu}$ to give

$$\Box h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{\alpha}{4\alpha+1}\eta_{\mu\nu}\bar{h} - \frac{\beta}{4\alpha+1}\eta_{\mu\nu}R^{(1)}, \tag{13}$$

and

$$\Box h = \frac{1}{4\alpha+1}\bar{h} - \frac{4\beta}{4\alpha+1}R^{(1)}. \tag{14}$$

Inserting Eqs. (13) and (14) into $\Box R^{(1)}_{\mu\nu}$ yields

$$\begin{aligned}\Box R^{(1)}_{\mu\nu} = \frac{1}{2}\Big[&\Box\bar{h}_{\mu\nu} - \frac{\alpha\eta_{\mu\nu}}{4\alpha+1}\Box\bar{h} - \partial_\mu\partial^\lambda\bar{h}_{\lambda\nu} - \partial_\nu\partial^\lambda\bar{h}_{\lambda\mu} \\ &+ \frac{2\alpha+1}{4\alpha+1}\partial_\mu\partial_\nu\bar{h} - \frac{2\beta}{4\alpha+1}\partial_\mu\partial_\nu R^{(1)} - \frac{\beta\eta_{\mu\nu}}{4\alpha+1}\Box R^{(1)}\Big].\end{aligned} \tag{15}$$