Physics Letters B 783 (2018) 36–42

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

On the thermodynamic phase structure of conformal gravity

Hao Xu [a,*], Man-Hong Yung [b,c]

[a] Department of Physics, Southern University of Science and Technology, Shenzhen 518055, China
[b] Institute for Quantum Science and Engineering and Department of Physics, Southern University of Science and Technology, Shenzhen 518055, China
[c] Shenzhen Key Laboratory of Quantum Science and Engineering, Shenzhen, 518055, China

ARTICLE INFO

Article history:
Received 5 May 2018
Received in revised form 14 June 2018
Accepted 15 June 2018
Available online 22 June 2018
Editor: M. Cvetič

ABSTRACT

The solutions of the Einstein equation are a subset of the solutions of conformal (Weyl) gravity, but the difference from the action means that the black hole thermodynamics of the two gravity theories would be different. In this paper we explore the thermodynamic phase structure for the conformal gravity in the four-dimensional AdS space–time. Special emphasis is put on the dependence on the parameter c_1 in the linear-r term in the metric. The thermodynamic phase structure of the conformal gravity is very rich, including two branches of equations of states, negative thermodynamic volume, zeroth-order phase transition, and Hawking–Page like phase transition.

Gravity theories containing higher-order derivative terms in the action are of fundamental interest for various reasons, in particular, generalizing Einstein gravity. One of the possible choices of such theory is the Lovelock gravity [1], where the action contains a sum of dimensionally-extended Euler densities and coincides with Einstein gravity in three and four dimensions. The integral of the k-th order term in Lovelock gravity gives the Euler character in dimension $d = 2k$, and the Einstein–Hilbert action is precisely the first-order term. The equations of motion, which depend only on the Riemann tensor and not on its derivatives, are also "Einstein-like". Furthermore, the higher-order derivative terms are also of interest in the context of the AdS/CFT correspondence [2,3]. They arise naturally in string theory [4].

Since the k-th order term in the Lovelock gravity only affects the local geometry of the manifold in $d \geq 2k + 1$, all the higher-order terms ($k \geq 2$) do not change the Einstein manifold in the lower dimensions. However, the higher-order terms do play an important role in the renormalization theory [5]. One of the gravity theories, known as the conformal (Weyl) gravity, which is described by a pure Weyl squared action, has been shown to be perturbatively renormalizable in four dimensions, although the massive modes are ghostlike. It is a local gauge theory of the conformal group, so the equation of motion determines the metric only up to an arbitrary conformal factor. The solutions of the Einstein equation are a subset of the solutions of conformal gravity; any space–time that is conformal to an Einstein space arises naturally as a solution of conformal gravity. In 2011, Maldacena demonstrated a remarkable result showing that conformal gravity with a Neumann boundary condition can select the Einstein solution out of conformal gravity [6,7].

Conformal gravity is also of interest for cosmology. Although Einstein gravity can well describe the physics within the solar system, such as the gravitational bending of light and the precession of the perihelion of Mercury, there are still some puzzles left on scales far beyond the distances of the Solar system. For example, the galactic rotation curves are not consistent with the predictions of Einstein's gravity; the unknown "dark matter", which interacts with the baryonic matter only via gravity or via the weak force, is introduced to fix the problem. In addition, the concept of dark energy is also introduced, which is much larger than the zero-point energy of the matter fields, in order to provide the energy source to explain the observation of our accelerating universe.

Alternatively, one may also question whether it is possible to modify the gravity theory in order to explain the physics at a large scale, while maintaining the behavior at the scale of the Solar system. Since conformal gravity possesses more solutions than Einstein gravity, the dark matter, dark energy problems can be fixed within the gravity theory [8–11], which motivates us to investigate conformal gravity in more detail. In particular, although conformal gravity and Einstein gravity may share the same space–time solutions, the difference from the action means that the black hole thermodynamics of the two gravity theories would be different, which points to an area where extensive research should be carried out.

* Corresponding author.
E-mail address: xuh5@sustc.edu.cn (H. Xu).

https://doi.org/10.1016/j.physletb.2018.06.045

Here, we focus on the thermodynamic phase structure of the conformal gravity in the four-dimensional AdS space–time. Moreover, we do not assume the "physical constants", such as Yukawa coupling constants, cosmological constant, and Lovelock coefficients, to be fixed values; they may be dynamical variables resulting form the vacuum expectation values; it is therefore reasonable to include them into the thermodynamic laws [12–16]. Recently, the idea of including the cosmological constant in the first law of black hole thermodynamics becomes popular. See, e.g., [17–35] for references and reviews. In particular, in AdS space–time the negative cosmological constant behaves like the pressure, while its conjugate variable can be considered as a thermodynamic volume. However, in Einstein gravity, the cosmological constant Λ comes from the action, thus varying Λ requires changing the system. Remarkably, one does not have to worry about this problem in conformal gravity, as Λ arises as the integral constant of the solution [36–38] instead of the action. This makes the analysis for the phase structure self-contained.

We start by giving a brief review of the black hole thermodynamics of conformal gravity [36]. First, the action is given by a square of the Weyl tensor,

$$S=\alpha\int \mathrm{d}^4x\sqrt{-g}C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}\ . \tag{1}$$

The coupling constant α plays an important role in critical gravity [39–41]. However, here it is independent of the equation of motion and do not change any qualitative feature of the thermodynamic quantities. Without loss of generality, we set $\alpha=1$ in the following. Moreover, the equation of motion is fourth order,

$$(2\nabla^{\rho}\nabla^{\sigma}+R^{\rho\sigma})C_{\mu\rho\sigma\nu}=0\,. \tag{2}$$

The most general spherical black hole solution for conformal gravity takes the following form [42,43]

$$\mathrm{d}s^2=-f(r)\mathrm{d}t^2+\frac{\mathrm{d}r^2}{f(r)}+r^2\mathrm{d}\Omega^2_{S^2}, \tag{3}$$

where $\mathrm{d}\Omega^2_{S^2}$ is the line element of a 2-dimensional sphere and

$$f(r)=c_0+c_1r+\frac{d}{r}-\frac{1}{3}\Lambda r^2. \tag{4}$$

Due to the conformal symmetry of the action, the Weyl rescaling of the above metric remains a static spherically symmetric solution, so the Birkhoff's theorem only restricts the static spherically symmetric solutions to a conformal class. There are four different integral constants, c_0, c_1, d, Λ in the metric. Three of them must obey a constraint,

$$c_0^2=3c_1d+1. \tag{5}$$

When $c_1=0$, this solution reduces to the well-known Schwarzschild (A)dS space–time. Notice that there exists a discrete freedom in choosing the constant c_0,

$$c_0=\pm\sqrt{3c_1d+1}. \tag{6}$$

Mathematically, we can also solve c_1 in terms of c_0 and d. However, from the thermodynamical perspective, it is preferable to take c_1 as independent thermodynamical parameter [36].

In our case, Λ plays the role of the cosmological constant. It is an integral constant but not from the action. The energy, which is defined by the conserved charge of the timelike killing vector, should be identified as enthalpy H of the system [36]

$$H=\frac{(c_1c_0-c_1-16\pi Pd)}{12\pi}. \tag{7}$$

The other thermodynamic quantities can also be obtained. The temperature is proportional to the surface gravity at the horizon with radius r_0

$$T=\frac{8\pi Pr_0^3-3c_0r_0-6d}{12\pi r_0{}^2}, \tag{8}$$

where r_0 is the largest root of $f(r_0)=0$ in AdS space–time, and its conjugate, i.e. the entropy is [36]

$$S=\frac{(r_0-c_0r_0-3d)}{3r_0}, \tag{9}$$

which is a function of c_0, r_0 and d rather than being proportional to the area of the horizon. If we also treat c_1 as a variable, its conjugate quantity is

$$\Psi=\frac{(c_0-1)}{12\pi}. \tag{10}$$

We take the cosmological constant Λ as the pressure,

$$P=-\frac{\Lambda}{8\pi}, \tag{11}$$

then the thermodynamic volume is

$$V=\left(\frac{\partial H}{\partial P}\right)_{S,c_1}=-\frac{2d}{3}. \tag{12}$$

So we can get the first law of black hole thermodynamics in conformal gravity

$$\mathrm{d}H=T\mathrm{d}S+\Psi\mathrm{d}c_1+V\,\mathrm{d}P \tag{13}$$

and the Smarr relation [36]

$$H=2PV+\Psi c_1. \tag{14}$$

The Gibbs free energy can be obtained by the relation (or by using the Euclidean action)

$$G=H-TS=\frac{2(c_0-1)r_0+(3+8\pi Pr_0^2)d}{12\pi r_0^2}. \tag{15}$$

Notice that here we have fixed the parameter α, which has dimensions of [length]^2. If we consider the contributions of the α, we will have $H\to\alpha H$, $S\to\alpha S$, $V\to\alpha V$, $\Psi\to\alpha\Psi$, $G\to\alpha G$ and obtain the ordinary scaling dimensions of these thermodynamic quantities [36]. The α does not change any qualitative feature of the thermodynamic quantities or the Smarr relation.

To study the phase structure of conformal gravity, we should begin with the equations of state in $P-V$ plane. By using $f(r_0)=0$, (5), and (8) to eliminate other unnecessary coefficients, we have two solutions of $P(T,r_0)$ and $V(T,r_0)$. The first one is

$$\begin{aligned}P_1&=\frac{T}{2r_0}-\frac{c_1r_0-\sqrt{1-4\pi Tc_1r_0^2}}{8\pi r_0{}^2},\\ V_1&=\frac{2r_0}{9}\left(4\pi Tr_0-c_1r_0-2\sqrt{1-4\pi Tc_1r_0^2}\right),\end{aligned} \tag{16}$$

while the other is

$$\begin{aligned}P_2&=\frac{T}{2r_0}-\frac{c_1r_0+\sqrt{1-4\pi Tc_1r_0^2}}{8\pi r_0{}^2},\\ V_2&=\frac{2r_0}{9}\left(4\pi Tr_0-c_1r_0+2\sqrt{1-4\pi Tc_1r_0^2}\right).\end{aligned} \tag{17}$$