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Solitons in a cavity for the Einstein-$SU(2)$ Non-linear Sigma Model and Skyrme model

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ABSTRACT

In this work, taking advantage of the Generalized Hedgehog Ansatz, we construct new self-gravitating solitons in a cavity with mirror-like boundary conditions for the $SU(2)$ Non-linear Sigma Model and Skyrme model. For spherically symmetric spacetimes, we are able to reduce the system to three independent equations that are numerically integrated. There are two branches of well-behaved solutions. The first branch is defined for arbitrary values of the Skyrme coupling and therefore also leads to a gravitating soliton in the Non-linear Sigma Model, while the second branch exists only for non-vanishing Skyrme coupling. The solutions are static and in the first branch are characterized by two integration constants that correspond to the frequency of the phase of the Skyrme field and the value of the Skyrme profile at the origin, while in the second branch the latter is the unique parameter characterizing the solutions. These parameters determine the size of the cavity, the redshift at the boundary of the cavity, the energy of the scalar field and the charge associated to a $U(1)$ global symmetry. We also show that within this ansatz, assuming analyticity of the matter fields, there are no spherically symmetric black hole solutions.

1. Introduction

Non-linear Sigma models appear in many contexts, as for example to describe the dynamics of Goldstone bosons [1], in condensed matter systems [2], in supergravity [3], as well as being the building blocks of classical string theory. In the case of light mesons, it can be shown that the low energy dynamics can be correctly described by a Non-linear Sigma Model for $SU(2)$. In such low energy processes, the mesons can be seen as Goldstone bosons. In flat spacetime, the inclusion of the Skyrme term allows to construct static regular solitons with finite energy, which describe baryons [4]. In the latter scenario the ansatz for the $SU(2)$ group element is given by $U_{\text{sol}} = \exp(iF(r)\vec{\tau}\cdot\hat{x})$, with $\vec{\tau}$ the $SU(2)$ generators. A more general ansatz is defined by the Generalized Hedgehog Ansatz, which includes U_{sol} as a particular case, and is defined by

$$U^{\pm 1} = Y^0\mathbf{1} \pm Y^i t_i\ , \tag{1}$$

where $\mathbf{1}$ is the 2×2 identity matrix, $t_i = -i\sigma_i$ the $SU(2)$ generators, σ_i being the Pauli matrices and

$$Y^0 = \cos\alpha(x^\mu)\ ,\quad Y^i = \widehat{n}^i \sin\alpha(x^\mu)\ ,\qquad (Y^0)^2 + Y_iY^i = 1\ , \tag{2}$$

with a generalized radial unit vector

$$\widehat{n}^1 = \cos\Theta(x^\mu)\sin F(x^\mu)\ ,\quad \widehat{n}^2 = \sin\Theta(x^\mu)\sin F(x^\mu)\ ,$$
$$\widehat{n}^3 = \cos F(x^\mu)\ . \tag{3}$$

Here α, Θ and F are arbitrary functions of the space–time coordinates. This ansatz was originally introduced in the context of the Gribov problem in regions with non-trivial topology [5], and has been shown to provide a very fruitful arena to construct new solutions of the theory. In reference [6], the compatibility of this ansatz on the Einstein–Skyrme theory was thoroughly explored considering a space–time which is a warped product of a two-dimensional space–time with an Euclidean constant curvature manifold. Also, within this ansatz, a novel non-linear superposition law was found in [7] for the Skyrme theory, which was latter extended to the curved geometry of $AdS_2\times S^2$ in reference [8]. Even more, the ansatz allows for exact solitons with a kink profile [9]. Asymptotically AdS wormholes and bouncing cosmologies with self-gravitating Skyrmions were constructed in [10] as well as

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other time dependent cosmological solutions with non-vanishing topological charge [11]. Also within the context of the generalized hedgehog ansatz, for the $SU(2)$ Non-linear Sigma Model, topologically non-trivial gravitating solutions were constructed in [12] which cannot decay on the trivial vacuum due to topological obstructions and, more recently, planar asymptotically AdS hairy black hole solutions were found in [13].

In this paper we will explore a new family of solutions within the Generalized Hedgehog Ansatz which describe spherically symmetric, static configurations in a cavity. By imposing mirror-like boundary condition for the matter field we numerically construct new self-gravitating solitons for the $SU(2)$ Skyrme model and Non-linear Sigma Model. In Section 2 we introduce the Generalized Hedgehog Ansatz. In Section 3 we reduce the system to three non-linear equations and argue that in order to have configurations with finite energy it is necessary to introduce a mirror at a finite proper distance from the origin. Section 4 is devoted to the numerical integration of the system that leads to two well-behaved branches. The first branch is well behaved for arbitrary values of the coupling constant of the Skyrme term λ, while the second leads to well-behaved solutions only for non-vanishing λ. Section 5 contain the conclusions and further comments as well as the proof that, within this ansatz, there are no black holes supported by an analytic Skyrme field.

2. The $SU(2)$ Einstein–Skyrme and Einstein-Nonlinear Sigma Model

In this paper we will be concerned with the gravitating Einstein–Skyrme model as well as with the Einstein-Non-linear Sigma Model systems. The action is given by

$$I[g,U]=\int d^4x\sqrt{-g}\left(\frac{R}{2\kappa}+\frac{K}{4}\mathrm{Tr}\left(A^{\mu}A_{\mu}+\frac{\lambda}{8}F_{\mu\nu}F^{\mu\nu}\right)\right), \quad (4)$$

where R is the Ricci scalar, $A_{\mu}:=U^{-1}\nabla_{\mu}U$ and $F_{\mu\nu}=[A_{\mu},A_{\nu}]$. Here U is a scalar field valued in $SU(2)$ and therefore $A_{\mu}=A^{i}_{\mu}t_i$. We work in the mostly plus signature, Greek and Latin indices run over spacetime and the algebra, respectively. Hereafter without loosing generality we set $K=1$.

The field equations for this theory are the Einstein equations

$$\mathcal{E}_{\mu\nu}=G_{\mu\nu}-T_{\mu\nu}=0\,, \quad (5)$$

with energy-momentum tensor given by

$$T_{\mu\nu}=-\frac{1}{2}\mathrm{Tr}\left(A_{\mu}A_{\nu}-\frac{1}{2}g_{\mu\nu}A_{\alpha}A^{\alpha}+\frac{\lambda}{4}(g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}-\frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta})\right),$$

satisfying the dominant energy condition [14], and the Skyrme equations

$$\nabla^{\mu}A_{\mu}+\frac{\lambda}{4}\nabla^{\mu}[A^{\nu},F_{\mu\nu}]=0. \quad (6)$$

We will consider the generalized hedgehog ansatz (2) and (3) with $F\left(x^{\mu}\right)=\frac{\pi}{2}$. The functions α and Θ of the ansatz (2) and (3) are scalar functions: α describes the energy profile of the configuration while Θ describes its orientation in isospin space. One can check that the above ansatz has vanishing baryon charge, thus we are within the pionic sector. The group manifold of $SU(2)$ is the three-sphere S^3, and our ansatz turns on the field along the $S^2\subset S^3$ submanifold. The advantage of the Generalized Hedgehog Ansatz is given by the fact that the Skyrme equations reduce to a single equation provided [6],

$$\Box\Theta=0\,,\ \nabla_{\mu}\Theta\nabla^{\mu}\alpha=0\,, \quad (7)$$

$$(\nabla^{\mu}\nabla^{\nu}\Theta)\nabla_{\mu}\Theta\nabla_{\nu}\Theta=0\,, \quad (8)$$

$$(\nabla^{\mu}\nabla^{\nu}\alpha)\nabla_{\mu}\alpha\nabla_{\nu}\Theta=0\,. \quad (9)$$

Even though these equations may seem too restrictive, we will show below that they are compatible with the existence of solitonic solutions in a cavity.

With this, the Einstein and Skyrme equations reduce to

$$\mathcal{E}_{\mu\nu}=G_{\mu\nu}-\kappa T_{\mu\nu}=0\,, \quad (10)$$

with

$$T_{\mu\nu}=\Bigg[(\nabla_{\mu}\alpha)(\nabla_{\nu}\alpha)+\sin^2\alpha(\nabla_{\mu}\Theta)(\nabla_{\nu}\Theta)+\lambda\sin^2\alpha\times\left((\nabla\Theta)^2(\nabla_{\mu}\alpha)(\nabla_{\nu}\alpha)+(\nabla\alpha)^2(\nabla_{\mu}\Theta)(\nabla_{\nu}\Theta)\right)-\frac{1}{2}g_{\mu\nu}\left((\nabla\alpha)^2+\sin^2\alpha(\nabla\Theta)^2+\lambda\sin^2\alpha(\nabla\Theta)^2(\nabla\alpha)^2\right)\Bigg], \quad (11)$$

supplemented by

$$\Box\alpha-\frac{1}{2}\sin(2\alpha)(\nabla\Theta)^2+\lambda\Bigg[(\nabla_{\mu}\alpha)\nabla^{\mu}\left(\sin^2\alpha(\nabla\Theta)^2\right)+\sin^2\alpha(\nabla\Theta)^2(\Box\alpha)-\frac{1}{2}\sin(2\alpha)(\nabla\alpha)^2(\nabla\Theta)^2\Bigg]=0\,. \quad (12)$$

These equations can also be obtained from the effective action

$$I_{\mathrm{eff}}=\int\sqrt{-g}\Bigg[\frac{R}{2\kappa}-2\Bigg(\partial_{\rho}\alpha\partial^{\rho}\alpha+\sin^2(\alpha)\partial_{\rho}\Theta\partial^{\rho}\Theta+\frac{\lambda}{2}\sin^2(\alpha)(\nabla\alpha)^2(\nabla\Theta)^2\Bigg)\Bigg], \quad (13)$$

provided the constraints (7)–(9) are fulfilled. Einstein equations (10) and the Skyrme equation (12) are obtained from the variation of I_{eff} w.r.t. the metric and the scalar α, respectively, and the equation for Θ is trivially satisfied after imposing the constraints (7)–(9). The effective action, as well as the constraints, are invariant under the global transformation

$$\delta_{(1)}\alpha=0\,,\qquad \delta_{(1)}\Theta=\epsilon\,, \quad (14)$$

where ϵ is a parameter. The symmetry transformation $\delta_{(1)}$ allows to construct a locally conserved current which, when integrated within the cavity, leads to a finite conserved charge.

3. The system and its finite energy solutions

We consider a static spherically symmetric space–time metric

$$ds^2=-f(r)dt^2+\frac{dr^2}{h(r)}+r^2\left(d\theta^2+\sin^2\theta d\phi^2\right)\,, \quad (15)$$

and the following dependence for the matter fields

$$\alpha=\alpha(r)\,,\qquad \Theta=\omega t\,, \quad (16)$$

where ω is a frequency, leading to a time independent energy-momentum tensor and therefore the physical quantities for this