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Coloured Alexander polynomials and KP hierarchy

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ABSTRACT

We discuss the relation between knot polynomials and the KP hierarchy. Mainly, we study the scaling 1-hook property of the coloured Alexander polynomial: $\mathcal{A}_R^{\mathcal{K}}(q) = \mathcal{A}_{[1]}^{\mathcal{K}}(q^{|R|})$ for all 1-hook Young diagrams R. Via the Kontsevich construction, it is reformulated as a system of linear equations. It appears that the solutions of this system induce the KP equations in the Hirota form. The Alexander polynomial is a specialization of the HOMFLY polynomial, and it is a kind of a dual to the double scaling limit, which gives the special polynomial, in the sense that, while the special polynomials provide solutions to the KP hierarchy, the Alexander polynomials provide the equations of this hierarchy. This gives a new connection with integrable properties of knot polynomials and puts an interesting question about the way the KP hierarchy is encoded in the full HOMFLY polynomial.

1. Introduction

Nowadays knot theory is of great interest in mathematical physics. This is due to the fact that knot invariants appear in various physical problems such as quantum field theories [1–3], quantum groups [4], lattice models [5], CFT [6], topological strings [7], quantum computing [8] etc. These correspondences lead to generalizations of some already known invariants and to discoveries of new ones.

The class of polynomial invariants is probably the most developed and the most actively studied. One of the most important in this class is the (unreduced) coloured HOMFLY polynomial $\mathcal{H}_R^{\mathcal{K}}(q,a)$ of the knot $\mathcal{K}$ coloured with representation R. It takes values in the ring of Laurent polynomials of two variables $\mathbb{Z}[q,q^{-1},a,a^{-1}]$. It may be defined as the vacuum expectation value of a Wilson loop along the knot in Chern–Simons gauge theory, with the gauge group $G = SU(N)$ and representation R [1,9]:

$$\mathcal{H}_R^{\mathcal{K}}(q,a) = \frac{1}{Z}\int DA\ e^{-\frac{i}{\hbar}S_{CS}[A]}\,W_R(K,A), \tag{1}$$

where the Wilson loop and the Chern–Simons action given by

$$\begin{aligned} W_R(K,A) &= \mathrm{tr}_R\ P\exp\left(\oint A_\mu^a(x)T^a dx^\mu\right), \\ S_{CS}[A] &= \frac{\kappa}{4\pi}\int_M \mathrm{Tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A). \end{aligned} \tag{2}$$

The variables q, a in the HOMFLY polynomial are

$$q = e^{\hbar}, \quad a = e^{N\hbar}, \quad \hbar := \frac{2\pi i}{\kappa + N}.$$

In such a parametrization, the polynomial may be represented as a series in the variable $\hbar$. Furthermore, there is a natural "quasi-classical" double scaling expansion given by $\hbar\to 0$, $N\to\infty$ such that $N\hbar$ stays fixed. In other words, this means taking $q = 1$ and keeping the variable a arbitrary. The polynomials that emerge as the special value of the HOMFLY polynomials $\mathcal{H}_R^{\mathcal{K}}(1,a) = \sigma_R^{\mathcal{K}}(a)$ are called the special polynomials [12]. Their R dependence has a simple power-like form, which expresses them through the special polynomial in the fundamental representation [12–15]:

$$\sigma_R^{\mathcal{K}}(a) = \left(\sigma_{[1]}^{\mathcal{K}}(a)\right)^{|R|}. \tag{3}$$

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Hereafter, we identify the representation R with the Young diagram associated with it: $R = \{R_i\}$, $R_1 \ge R_2 \ge \ldots \ge R_{l(R)}$, $|R| := \sum_i R_i$.

These polynomials exhibit integrable properties [15,16]: their R dependence allows one to construct from them a KP τ-function, which is the value of the Ooguri–Vafa partition function:

$$\mathcal{Z}^{\mathcal{K}}(\bar{p}|a,q) = \sum_R \mathcal{H}_R^{\mathcal{K}}(a,q) D_R \chi_R\{\bar{p}\}, \tag{4}$$

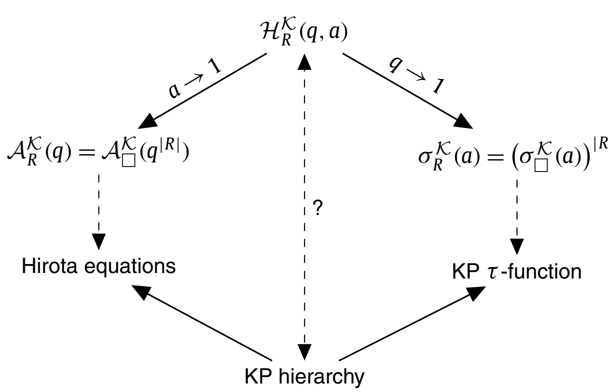
at $q = 1$. In this formula, $\chi_R\{p\}$ is the Schur polynomial in the representation R, $D_R = \chi_R\{p^*\}$ with $p_k^* = \frac{a^k - a^{-k}}{q^k - q^{-k}}$ is the quantum dimension.

On the other hand, the "dual" limit would be taking $a \to 1$ and leaving us with $\mathcal{H}_R^{\mathcal{K}}(q,1)$. For the fundamental representation R, this gives the oldest polynomial knot invariant, the Alexander polynomial. Its coloured version displays a "dual" property with respect to R [13,18][1]:

$$\mathcal{A}_R^{\mathcal{K}}(q) = \mathcal{A}_{[1]}^{\mathcal{K}}(q^{|R|}), \quad \text{where } R = [r, 1^L], \tag{5}$$

which holds only for the representations corresponding to 1-hook Young diagrams. We have studied this property perturbatively and discovered that it also miraculously related to the KP hierarchy. We found that, while the special polynomials provide solutions to the KP hierarchy, the Alexander polynomials induce the equations of the KP hierarchy. In this paper, we state this result giving the shorter half of the proof. Explicit calculations and the detailed proof will be presented elsewhere.

This observation not only gives another example of integrable properties of knot polynomials, but also argues in favour of use of the term "dual" in discussing the two limits of the HOMFLY polynomial. The results can be summarized in the following diagram:



Another well-known perturbative expansion of the HOMFLY polynomial is the loop expansion [9], which is based on the gauge invariance of Chern–Simons theory. Evaluating the Wilson loop correlators can be done in some fixed gauge. In the temporal gauge [10], $A_0 = 0$, the Wilson loop acquires the polynomial form of the coloured HOMFLY invariant. When calculated in the holomorphic gauge [11] $A_x + iA_y = 0$, it gives the Kontsevich integral [19,20]. The theory is gauge invariant, therefore the two object are equal, however, the Kontsevich integral is a perturbative expansion and the HOMFLY polynomial is not. Therefore this construction gives a perturbative description of the HOMFLY polynomial with arbitrary variables q, a. It appears to have a nicely looking structure [21]:

$$\mathcal{H}_R^{\mathcal{K}} = \sum_n \left(\sum_j v_{n,j}^{\mathcal{K}} r_{n,j}^R \right) \hbar^n. \tag{6}$$

A remarkable fact is that the knot dependence and the group theoretic dependence split explicitly. The group one is represented by the so called group factors $r_{n,j}^R$. They are group invariants that appear in the Kontsevich integral as some trivalent diagrams, which are further expressed as traces of products of the algebra generators T^a and the structure constants:

$$r_{n,j}^R \sim \mathrm{tr}_R(T^{a_1} \ldots T^{a_n}). \tag{7}$$

The knot dependent part is $v_{n,j}^{\mathcal{K}}$, which are some numerical invariants. Another point is that they appear to be exactly the famous Vassiliev invariants or invariants of finite type [21,22]. These numerical invariants are considered as potential candidates for a complete set of invariant and are therefore important to study. In this paper, however, we mostly focus on the group theoretic part.

In section 2, we describe the Alexander polynomial and its basic property, in section 3, we discuss a set of equations that originate from this basic property, and, in section 4, we review the Hirota bilinear identities and the KP hierarchy in order to compare them, in section 5, with the Alexander set of equations. In section 5, we demonstrate that the Hirota KP bilinear equations are satisfied when the Hirota derivatives are replaced by the Casimir eigenvalues in the 1-hook diagrams, and this solution is equivalent to the solution of the Alexander set of equations. This is our main result in this paper, while another our result is the number of actually different KP equations of each order which coincides with the number of solutions of the Alexander equations.

2. Alexander polynomial

The Alexander polynomial is a knot invariant in the ring of Laurent polynomials in one variable $\mathbb{Z}[q^{-1}, q]$. Originally, it is defined via the $H_1(X_\infty)$ homology group of the infinite cyclic cover of the knot complement $S^3 \setminus \mathcal{K}$ and is denoted as $\mathcal{A}(q)$ [23].

Apart from its pure topological construction, it also appears as a specific value of the ordinary (fundamental) HOMFLY polynomial [22]:

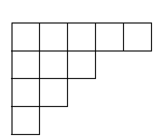
$$\mathcal{A}^{\mathcal{K}}(q) = \mathcal{H}^{\mathcal{K}}(q,1). \tag{8}$$

As higher representations of the gauge group lead to the coloured HOMFLY polynomials, one can immediately define the coloured Alexander polynomials

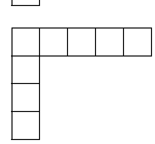
$$\mathcal{A}_R^{\mathcal{K}}(q) = \mathcal{H}_R^{\mathcal{K}}(q,1) \quad \text{or} \quad \mathcal{A}_R^{\mathcal{K}}(e^{\hbar}) = \lim_{N \to 0} \mathcal{H}_R^{\mathcal{K}}(e^{\hbar}, e^{N\hbar}). \tag{9}$$

As we already mentioned (5), this polynomial has a peculiar R dependence, but only for special representations.

A 1-hook Young diagram is a diagram of the form $\lambda = [r, \underbrace{1, \ldots, 1}_{L}]$:



[5, 3, 2, 1] diagram,



1-hook diagram [5, 1, 1, 1]

[1] Let us note that our notion of coloured Alexander polynomial is totally different from that defined in [17].