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Vacuum polarization energy of the Shifman–Voloshin soliton

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ABSTRACT

We compute the vacuum polarization energy of soliton configurations in a model with two scalar fields in one space dimension using spectral methods. The second field represents an extension of the conventional ϕ^4 kink soliton model. We find that the vacuum polarization energy destabilizes the soliton except when the fields have identical masses. In that case the model is equivalent to two independent ϕ^4 models.

1. Introduction

Two-field models supporting solitons in one space dimension obtainable as Bogomol'nyi–Prasad–Sommerfeld (BPS) solutions have been considered in the context of a number of applications, including supersymmetry and domain walls, see [1–7] and references therein. The essential feature leading to these applications is that in one space dimension the soliton has a localized kink shape, which becomes a surface (domain wall) when embedded in higher dimensions. When two (or more) fields interact multiple kinks at finite separation(s) emerge. The BPS construction is then carried out by writing a superpotential for the fields. The simplest such model has been introduced by Bazeia et al. [4] who also constructed some of its soliton solutions, while the full spectrum of solitons, including numerical simulations, was uncovered by Shifman and Voloshin [6]. In Ref. [8] the analytically known solitons of this model were considered as an illustration of general techniques allowing for the extension of scattering theory methods for computing one-loop quantum corrections [9] to the case of models with a mass gap, which then have multiple thresholds in the scattering problem. Such corrections were computed in that model for the simple cases where the soliton does not couple the fluctuation modes of the two fields in Ref. [10] and for small and moderate separation of the kinks in Refs. [11,12] using heat kernel methods [13].

In this Letter, we apply the methods of Ref. [8] to study these quantum corrections in more detail by going beyond the analytically known solitons for this particular model, which we define following the approach and conventions of Ref. [4]. We show that quantum corrections can significantly alter the classical stability of solitons in this model. In particular, the model can become unstable to the formation of a kink–antikink pair separated by a large region of *secondary* vacuum, whose classical energy density equals that of the *primary* vacuum outside the kink–antikink pair, but whose one-loop quantum energy density is negative.

The Bazeia model extends the ϕ^4 model by a second scalar field χ. Its Lagrangian reads

$$\mathcal{L}=\frac{1}{2}\left[\partial_\nu\phi\partial^\nu\phi+\partial_\nu\chi\partial^\nu\chi\right]-\frac{\lambda}{4}\left[\phi^2-\frac{M^2}{2\lambda}+\frac{\mu}{2}\chi^2\right]^2 -\frac{\lambda}{4}\mu^2\chi^2\phi^2\,. \qquad (1)$$

The Lagrangian contains the typical coupling constant λ and the mass scale M as in the conventional ϕ^4 model. We will discuss the meaning of the dimensionless coupling constant μ shortly. First we note that in the case $\mu=2$, when the two fields are indistinguishable, the orthogonal transformation $\varphi_{1,2}=\frac{1}{\sqrt{2}}[\chi\pm\phi]$ decouples the model into

$$\left[\phi^2-\frac{M^2}{2\lambda}+\chi^2\right]^2+\lambda\chi^2\phi^2=2\left[\varphi_1^2-\frac{M^2}{4\lambda}\right]^2+2\left[\varphi_2^2-\frac{M^2}{4\lambda}\right]^2,$$

which is a sum of two conventional and identical ϕ^4 models. As a result, the known results [15] from the ϕ^4 model with its kink soliton solution will provide checks of our calculations.

There are two distinct vacuum configurations. First, the solution with $\phi=\pm M/\sqrt{2\lambda}$ and $\chi=0$, adopted from the ϕ^4 model, and second, the solution with $\phi=0$ and $\chi=\pm M/\sqrt{\mu\lambda}$. Later we will see that only the first allows for BPS soliton solutions unless $\mu=2$, and thus we refer to it as the *primary* vacuum and the second as the *secondary* vacuum. The masses for fluctuations around

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the primary vacuum are $m_\phi = M$ and $m_\chi = \mu M/2$. That is, the dimensionless coupling constant is twice the ratio of the two masses.

After appropriate redefinition of the fields, $(\phi, \chi) \to (M/\sqrt{2\lambda})(\phi, \chi)$ and the coordinates, $x_\nu \to 2x_\nu/M$ the rescaled Lagrangian, $\mathcal{L} \to (M^4/8\lambda)\mathcal{L}$ is conveniently expressed as

$$\mathcal{L} = \frac{1}{2}\left[\partial_\nu\phi\partial^\nu\phi + \partial_\nu\chi\partial^\nu\chi\right] - U(\phi,\chi) \quad \text{with}$$
$$U(\phi,\chi) = \frac{1}{2}\left[\phi^2 - 1 + \frac{\mu}{2}\chi^2\right]^2 + \frac{\mu^2}{2}\phi^2\chi^2 \,. \tag{2}$$

In these units the primary vacuum configuration is $\phi_{\rm vac} = \pm 1$ and $\chi_{\rm vac} = 0$ so that $m_\chi = \mu$ and $m_\phi = 2$. Note that with these dimensionless variables the classical mass is measured in units of M^3/λ, while the one-loop quantum energy, which is central to the current study, scales with M/m_ϕ. The different scales arise from the overall loop-counting factor in $\mathcal{L}$ that emerges from canonical quantization.

In Section 2 we describe the construction of the solitons in this model. Following, in Section 3, we review the computation of the one-loop quantum, or vacuum polarization energy (VPE) in the no-tadpole renormalization scheme. In Section 4 we present the numerical results for the VPE and show that it produces an instability unless $\mu = 2$. We conclude in Section 5. In an Appendix we show that the finite renormalization imposing on-shell conditions does not alter the conclusion of instability.

2. Soliton

The Bazeia model [4] is defined to allow a BPS construction for the classical energy

$$E_{\rm cl} = \frac{1}{2}\int_{-\infty}^{\infty} dx \left[\phi'^2 + \chi'^2 + \left(\phi^2 - 1 + \frac{\mu}{2}\chi^2\right)^2 + \mu^2\phi^2\chi^2\right]$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} dx \left[\left(\phi^2 - 1 + \frac{\mu}{2}\chi^2 \pm \phi'\right)^2 + \left(\mu\phi\chi \pm \chi'\right)^2\right]$$
$$\pm\left[\phi - \frac{1}{3}\phi^3 - \mu\phi\chi^2\right]_{-\infty}^{\infty} \,, \tag{3}$$

where the prime denotes the derivative with respect to the (dimensionless) coordinate x. We immediately see that only profile functions that assume the primary vacuum configuration can have finite non-zero energy.[1] Choosing $\phi(\pm\infty) = \pm 1$ requires the upper sign in Eq. (3) because $\phi(x)$ must (monotonically) increase. Then the BPS equations read

$$\frac{d\chi(x)}{dx} = -\mu\phi(x)\chi(x) \quad \text{and}$$
$$\frac{d\phi(x)}{dx} = 1 - \phi^2(x) - \frac{\mu}{2}\chi^2(x) \,. \tag{4}$$

These coupled differential equations have been studied in detail by Shifman and Voloshin [6]. For completeness we discuss those results. The model exhibits translational invariance and we center the (eventual) soliton at $x_0 = 0$. Then χ and ϕ are symmetric and anti-symmetric functions, respectively,[2] and so $\phi(0) = 0$ and $\chi'(0) = 0$. We are free to choose $\chi(0) \ge 0$. If $\chi(0) > \sqrt{2/\mu}$, $\phi'(0) < 0$ so that $\phi(0^+) < 0$. In turn χ would increase and $\chi(0)$

Table I
Analytically known soliton solutions [4,6].

	$\phi(x)$	$\chi(x)$	Parameters
I)	$\tanh(x)$	0	$a = 0$
II)	$\tanh(\mu x)$	$\frac{\sqrt{2(1/\mu - 1)}}{\cosh(\mu x)}$	$\mu < 1$, $a = \sqrt{1-\mu}$
III)	$\frac{\sinh(2x)}{b + \cosh(2x)}$	$\frac{\sqrt{b^2-1}}{b+\cosh(2x)}$	$\mu = 2$, $b = \frac{1+a^2}{1-a^2}$
IV)	$\frac{(1-a^2)\sinh(x)}{a^2 + (1-a^2)\cosh(x)}$	$\frac{2a}{\sqrt{a^2+(1-a^2)\cosh(x)}}$	$\mu = \frac{1}{2}$

would be a minimum. Furthermore ϕ' would turn even more negative and not approach $+1$ at spatial infinity. By contradiction we thus conclude that $\sqrt{2/\mu}$ is an upper bound for $\chi(0)$ and we parameterize $\chi(0) = a\sqrt{2/\mu}$ with $0 \le a < 1$. An equivalent bound was derived in Ref. [6] from the condition that the solution to

$$\frac{d\phi^2}{d\chi} = 2\phi\frac{d\phi}{dx}\left(\frac{d\chi}{dx}\right)^{-1} = -\frac{2 - 2\phi^2 - \mu\chi^2}{\mu\chi}$$

is consistent with $\phi^2 \ge 0$.

Because of the reflection symmetry $x \leftrightarrow -x$ it is sufficient to solve Eqs. (4) on the half-line $x \ge 0$. In the numerical simulation we initialize $\phi(0) = 0$ and $\chi(0) = a\sqrt{2/\mu}$ and vary a. For any numerical solution we then verify that the first integral in Eq. (3) produces $E_{\rm cl} = \frac{4}{3}$. We also verify that the numerical solutions agree with the analytically known results listed in Table I.

We thus find that the various known analytical solutions are not independent but are related by a single parameter. If these solitons were independent, a third zero mode for the small amplitude fluctuations about the soliton along the direction in field space connecting the solutions would have emerged, but only two have been observed [8]. Stated otherwise, the solitons are parameterized by two continuous parameters [6]: the center of the soliton, which we set to zero, and the amplitude of the χ field, which we parameterize by a. Varying these parameters produces the two observed zero modes. Alternatively, the family of solitons can be constructed by successively adding infinitesimal contributions proportional to the zero mode wave-function.

The limit $a \to 1$ deserves further discussion. In that case, the right-hand-sides of Eq. (4) are tiny in a wide region around $x = 0$, so that the profiles stay constant at their $x = 0$ values. Eventually two well separated structures emerge at which ϕ changes from -1 to zero and zero to $+1$, respectively [6]. Simultaneously, χ changes from zero to $a\sqrt{2/\mu}$ and back to zero. We show this behavior in Fig. 1 (where we only display the $x \ge 0$ regime since the profiles are obtained by reflection for $x \le 0$). When $a \to 1$, $\chi(0)$ approaches $\sqrt{2/\mu}$ and the slope $\phi'(0)$ decreases so that the profiles assume the secondary vacuum configuration in a large range of coordinate space.

While μ is a model parameter, a is a variational parameter that we tune to minimize the total energy. Since $E_{\rm cl}$ does not depend on a, we only need to consider the a dependence of the VPE, whose formulation we discuss next.

3. Vacuum polarization energy

The computation of the vacuum polarization energy in models with a mass gap ($\mu \ne 2$) has been established in Ref. [8]. We briefly summarize it here. The central input is the Jost function for imaginary momenta $t = ik$. The starting point for its computation is the second order differential equation

$$Z''(t,x) = 2Z'(t,x)D(t) + M^2 Z(t,x) - Z(t,x)M^2 + V(x)Z(t,x)$$
$$\text{with} \quad M^2 = \begin{pmatrix} \mu^2 & 0 \\ 0 & 4 \end{pmatrix} \tag{5}$$

[1] For $\mu = 2$ an alternative BPS construction is possible producing a soliton with $\lim_{|x|\to\infty}\chi(x) \ne 0$.

[2] Eqs. (4) also allow the opposite choice; but then the energy, Eq. (3) is zero.